Game Theory I

A Strategic Situation (due to Ben Polak)

Player 2

Player 1

	α	β
α	B-, B-	A, C
β	C, A	A-, A-

Selfish Students

		Selfish 2	
		α	β
Selfish 1	α	$1, 1$	$3, 0$
	β	$0, 3$	$2, 2$

Selfish Students

		Selfish 2	
		α	β
Selfish 1	α	$1, 1$	$3, 0$
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- No matter what Selfish 2 does, Selfish 1 wants to choose α (and vice versa)

Selfish Students

		Selfish 2	
		α	β
Selfish 1	α	$1, 1$	$3, 0$
	β	$0, 3$	$2, 2$

- No matter what Selfish 2 does, Selfish 1 wants to choose α (and vice versa)
- (α, α) is a sensible prediction for what will happen

Nice Students

		Nice 2	
		α	β
Nice 1	α	$2, 2$	$1, 0$
	β	$0, 1$	$3, 3$

Nice Students

		Nice 2	
		α	β
Nice 1	α	$2, 2$	$1, 0$
	β	$0, 1$	$3, 3$

- Each nice student wants to match the behavior of the other nice student

Nice Students

		Nice 2	
		α	β
Nice 1	α	$2, 2$	$1, 0$
	β	$0, 1$	$3, 3$

- Each nice student wants to match the behavior of the other nice student
- (α, α) or (β, β) seem sensible.

Nice Students

		Nice 2	
		α	β
Nice 1	α	$2, 2$	$1, 0$
	β	$0, 1$	$3, 3$

- Each nice student wants to match the behavior of the other nice student
- (α, α) or (β, β) seem sensible.
- We need to know what people think about each other's behavior to have a prediction

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1, 2$	$3, 0$
	β	$0, 1$	$2, 3$

Selfish vs. Nice

		Nice α	Nice β
Selfish	α	$1, 2$	$3, 0$
	β	$0, 1$	$2, 3$

- Nice wants to match what Selfish does

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1, 2$	$3, 0$
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- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to player α

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- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to player α
- If Nice can think one step about Selfish, she should realize she should play α

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- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to player α
- If Nice can think one step about Selfish, she should realize she should play α
- (α, α) seems the sensible prediction

Outline

Strategic Form Games

Solving a Game: Nash Equilibrium

Components of a Game

Players: Who is involved?

Strategies: What can they do?

Payoffs: What do they want?

Chicken

Player 2

Player 1		Straight	Swerve
	Straight	$0, 0$	$3, 1$
	Swerve	$1, 3$	$2, 2$

Choosing a Restaurant

		Rebecca	
		P	V
Ethan	P	$4, 3$	$1, 1$
	V	$0, 0$	$3, 4$

Working in a Team

2 players

Player i chooses effort $s_i \geq 0$

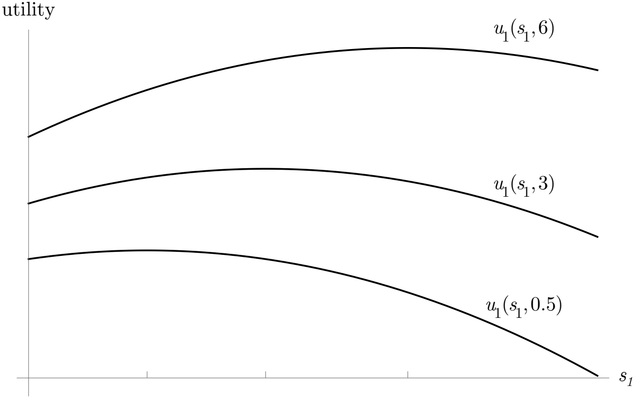
Jointly produce a product. Each enjoys an amount

$$\pi(s_1, s_2) = s_1 + s_2 + \frac{s_1 \times s_2}{2}$$

Cost of effort is s_i^2

$$u_i(s_1, s_2) = \pi(s_1, s_2) - s_i^2$$

Player 1's payoffs as a function of each player's strategy



Demand Bargaining

N players

Each player "demands" a real number in $[0, 10]$

If the demands sum to 10 or less, each player's payoff is her bid

Otherwise players' payoffs are 0

Outline

Strategic Form Games

Solving a Game: Nash Equilibrium

Nash Equilibrium

A strategy profile where no individual has a unilateral incentive to change her behavior

Before we talk about why this is our central solution concept, let's formalize it

Notation

Player i's strategy

- s_i

Set of all possible strategies for Player i

- S_i

Strategy profile (one strategy for each player)

- $\mathbf{s} = (s_1, s_2, \ldots, s_N)$

Strategy profile for all players except i

- $\mathbf{s_{-i}} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$

Different notation for strategy profile

- $\mathbf{s} = (s_i, \mathbf{s_{-i}})$

Selfish Students

		Player 2	
		α	β
Player 1	α	$1, 1$	$3, 0$
	β	$0, 3$	$2, 2$

$S_i = \{\alpha, \beta\}$

4 strategy profiles: $(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)$

Chicken

		Player 2	
		Straight	Swerve
Player 1	Straight	$0, 0$	$3, 1$
	Swerve	$1, 3$	$2, 2$

$S_i = \{\text{Straight, Swerve}\}$

4 strategy profiles: (Straight, Straight), (Straight, Swerve), (Swerve, Straight), (Swerve, Swerve)

Choosing a Restaurant

		Rebecca	
		P	V
Ethan	P	$4, 3$	$1, 1$
	V	$0, 0$	$3, 4$

$S_E = ?$ $\quad$ $S_R = ?$

Strategy profiles: ?

Demand Bargaining with 3 players

$S_i = [0, 10]$

- Player i can choose any real number between 0 and 10

$\mathbf{s} = (s_1 = 1, s_2 = 4, s_3 = 7) = (1, 4, 7)$

- An example of a strategy profile

$\mathbf{s_{-2}} = (1, 7)$

- Same strategy profile, with player 2's strategy omitted

$\mathbf{s} = (\mathbf{s_{-2}}, s_2) = ((1, 7), 4)$

- Reconstructing the strategy profile

Notating Payoffs

Players' payoffs are defined over strategy profiles

- A strategy profile implies an outcome of the game

Player i's payoff from the strategy profile $\mathbf{s}$ is

$$u_i(\mathbf{s})$$

Player i's payoff if she chooses s_i and others play as in $\mathbf{s_{-i}}$

$$u_i(s_i, \mathbf{s_{-i}})$$

Nash Equilibrium

Consider a game with N players. A strategy profile $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a **Nash equilibrium** of the game if, for every player i

$$u_i(s_i^*, \mathbf{s_{-i}}^*) \geq u_i(s_i', \mathbf{s_{-i}}^*)$$

for all $s_i' \in S_i$

Best Responses

A strategy, s_i, is a **best response** by Player i to a profile of strategies for all other players, $\mathbf{s_{-i}}$, if

$$u_i(s_i, \mathbf{s_{-i}}) \geq u_i(s_i', \mathbf{s_{-i}})$$

for all $s_i' \in S_i$

Best Response Correspondence

Player i's **best response correspondence**, BR_i, is a mapping from strategies for all players other than i into subsets of S_i satisfying the following condition:

- For each $\mathbf{s_{-i}}$, the mapping yields a set of strategies for Player i, $\mathrm{BR}_i(\mathbf{s_{-i}})$, such that s_i is in $\mathrm{BR}_i(\mathbf{s_{-i}})$ if and only if s_i is a best response to $\mathbf{s_{-i}}$

An Equivalent Definition of NE

Consider a game with N players. A strategy profile $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a **Nash equilibrium** of the game if s_i^* is a best response to $\mathbf{s_{-i}}^*$ for each $i = 1, 2, \ldots, N$

Selfish vs. Nice

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Selfish vs. Nice

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Chicken

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Chicken

Player 2

Player 1

		Swerve
Straight	$0, 0$	$3^{\checkmark}, 1$
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Chicken

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Player 1	Straight	$0, 0$	$3^{\checkmark}, 1^{\checkmark}$
	Swerve	$1^{\checkmark}, 3^{\checkmark}$	$2, 2$

You Solve Choosing a Restaurant

			Rebecca	
			P	V
Ethan	P		4, 3	1, 1
	V		0, 0	3, 4

Another Practice Game

		Player 2	
		L	R
Player 1	U	$10, 2$	$3, 4$
	D	$-1, 0$	$5, 7$

Working in a Team

$$u_1(s_1, s_2) = \pi(s_1, s_2) - s_1^2 = s_1 + s_2 + \frac{s_1 s_2}{2} - s_1^2$$

Find Player i's best response by maximizing for each s_2

$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = 1 + \frac{s_2}{2} - 2s_1$$

Working in a Team

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First-order condition sets this equal to 0 to get $\mathrm{BR}_1(s_2)$

$$1 + \frac{s_2}{2} - 2\,\mathrm{BR}_1(s_2) = 0$$

Working in a Team

$$u_1(s_1, s_2) = \pi(s_1, s_2) - s_1^2 = s_1 + s_2 + \frac{s_1 s_2}{2} - s_1^2$$

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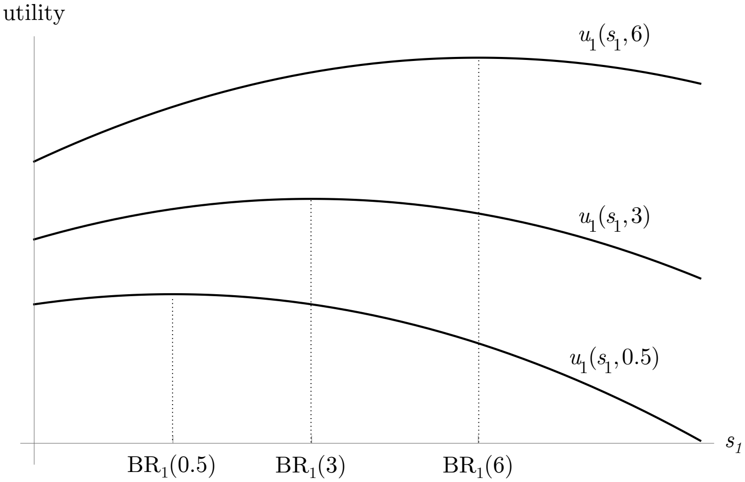
First-order condition sets this equal to 0 to get $\mathrm{BR}_1(s_2)$

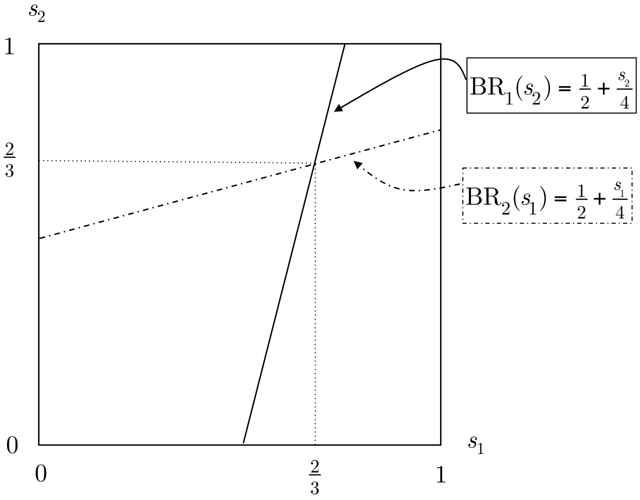
$$1 + \frac{s_2}{2} - 2\,\mathrm{BR}_1(s_2) = 0$$

$$\mathrm{BR}_1(s_2) = \frac{1}{2} + \frac{s_2}{4} \qquad \mathrm{BR}_2(s_1) = \frac{1}{2} + \frac{s_1}{4}$$

Player 1's Best Response



Nash Equilibrium



Solving for NE

Since best responses are unique, a NE is a profile, (s_1^*, s_2^*) satisfying

$$s_1^* = \mathrm{BR}_1(s_2^*) = \frac{1}{2} + \frac{s_2^*}{4} \qquad s_2^* = \mathrm{BR}_2(s_1^*) = \frac{1}{2} + \frac{s_1^*}{4}$$

Substituting

$$s_1^* = \frac{1}{2} + \frac{\frac{1}{2} + \frac{s_1^*}{4}}{4}$$

$$s_1^* = \frac{2}{3} \qquad s_2^* = \frac{2}{3}$$

Practice Game with Continuous Choices

2 players

Each player, i, chooses a real number s_i

There is a benefit of value 1 to be divided between the players

At a strategy profile (s_i, s_{-i}), Player i wins a share

$$\frac{s_i}{s_i + s_{-i}}$$

The cost of s_i is s_i

Solving

Write down Player 1's payoff from (s_1, s_2)

Solving

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$$u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1$$

Solving

Write down Player 1's payoff from (s_1, s_2)

$$u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1$$

Calculate Player 1's best response correspondence

Solving

Write down Player 1's payoff from (s_1, s_2)

$$u_1(s_1, s_2) = \frac{s_1}{s_1 + s_2} \times 1 - s_1$$

Calculate Player 1's best response correspondence

$$\frac{\partial u_1(s_1, s_2)}{\partial s_1} = \frac{s_1 + s_2 - s_1}{(s_1 + s_2)^2} \times 1 - 1 = \frac{s_2}{(s_1 + s_2)^2} - 1$$

Set equal to zero to maximize

$$\frac{s_2}{(\mathrm{BR}_1(s_2) + s_2)^2} - 1 = 0 \Rightarrow \mathrm{BR}_1(s_2) = \sqrt{s_2} - s_2$$

Solving^2

Player 2 is symmetric to Player 1, so write down both players' best response correspondences

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Player 2 is symmetric to Player 1, so write down both players' best response correspondences

$$\mathrm{BR}_1(s_2) = \sqrt{s_2} - s_2 \qquad \mathrm{BR}_2(s_1) = \sqrt{s_1} - s_1$$

Solving[2]

Player 2 is symmetric to Player 1, so write down both players' best response correspondences

$$\mathrm{BR}_1(s_2) = \sqrt{s_2} - s_2 \qquad \mathrm{BR}_2(s_1) = \sqrt{s_1} - s_1$$

At a NE each player is playing a best response to the other. Write down two equations that characterize equilibrium.

SOLVING[2]

Player 2 is symmetric to Player 1, so write down both players' best response correspondences

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At a NE each player is playing a best response to the other. Write down two equations that characterize equilibrium.

$$s_1^* = \sqrt{s_2^*} - s_2^* \qquad s_2^* = \sqrt{s_1^*} - s_1^*$$

Solving[3]

$$s_1^* = \sqrt{s_2^*} - s_2^* \qquad s_2^* = \sqrt{s_1^*} - s_1^*$$

Use substitution to find Player 1's equilibrium action

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$$s_1^* = \sqrt{s_2^*} - s_2^* \qquad s_2^* = \sqrt{s_1^*} - s_1^*$$

Use substitution to find Player 1's equilibrium action

$$s_1^* = \sqrt{\sqrt{s_1^*} - s_1^*} - \left(\sqrt{s_1^*} - s_1^*\right) \Rightarrow s_1^* = \frac{1}{4}$$

Now substitute this in to find Player 2's equilibrium action

Solving[3]

$$s_1^* = \sqrt{s_2^*} - s_2^* \qquad s_2^* = \sqrt{s_1^*} - s_1^*$$

Use substitution to find Player 1's equilibrium action

$$s_1^* = \sqrt{\sqrt{s_1^*} - s_1^*} - \left(\sqrt{s_1^*} - s_1^*\right) \Rightarrow s_1^* = \frac{1}{4}$$

Now substitute this in to find Player 2's equilibrium action

$$s_2^* = \sqrt{\frac{1}{4}} - \frac{1}{4} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Why Nash Equilibrium?

No regrets

Social learning

Self-enforcing agreements

Analyst humility

Take Aways

A Nash Equilibrium is a strategy profile where each player is best responding to what all other players are doing

You find a NE by calculating each player's best response correspondence and seeing where they intersect

NE is our main *solution concept* for strategic situations