Game Theory I

A Strategic Situation (due to Ben Polak)

Player 2

		α	β
Player 1	α	B-, B-	A, C
	β	C, A	A-, A-

Selfish Students

		Selfish 2	
		α	β
Selfish 1	α	$1, 1$	$3, 0$
	β	$0, 3$	$2, 2$

Selfish Students

		Selfish 2	
		α	β
Selfish 1	α	$1, 1$	$3, 0$
	β	$0, 3$	$2, 2$

- No matter what Selfish 2 does, Selfish 1 wants to choose α (and vice versa)

Selfish Students

		Selfish 2	
		α	β
Selfish 1	α	$1, 1$	$3, 0$
	β	$0, 3$	$2, 2$

- No matter what Selfish 2 does, Selfish 1 wants to choose α (and vice versa)
- (α, α) is a sensible prediction for what will happen

Nice Students

		Nice 2	
		α	β
Nice 1	α	$2, 2$	$1, 0$
	β	$0, 1$	$3, 3$

Nice Students

Nice 2

		α	β
Nice 1	α	$2, 2$	$1, 0$
	β	$0, 1$	$3, 3$

- Each nice student wants to match the behavior of the other nice student

Nice Students

		Nice 2	
		α	β
Nice 1	α	$2, 2$	$1, 0$
	β	$0, 1$	$3, 3$

- Each nice student wants to match the behavior of the other nice student
- (α, α) or (β, β) seem sensible.

Nice Students

		Nice 2	
		α	β
Nice 1	α	$2, 2$	$1, 0$
	β	$0, 1$	$3, 3$

- Each nice student wants to match the behavior of the other nice student
- (α, α) or (β, β) seem sensible.
- We need to know what people think about each other's behavior to have a prediction

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1, 2$	$3, 0$
	β	$0, 1$	$2, 3$

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1, 2$	$3, 0$
	β	$0, 1$	$2, 3$

- Nice wants to match what Selfish does

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1,2$	$3,0$
	β	$0,1$	$2,3$

- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to player α

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1, 2$	$3, 0$
	β	$0, 1$	$2, 3$

- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to player α
- If Nice can think one step about Selfish, she should realize she should play α

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1, 2$	$3, 0$
	β	$0, 1$	$2, 3$

- Nice wants to match what Selfish does
- No matter what Nice does, Selfish wants to player α
- If Nice can think one step about Selfish, she should realize she should play α
- (α, α) seems the sensible prediction

Outline

Strategic Form Games

Solving a Game: Nash Equilibrium

Components of a Game

Players: Who is involved?

Strategies: What can they do?

Payoffs: What do they want?

Chicken

		Player 2	
		Straight	Swerve
Player 1	Straight	$0, 0$	$3, 1$
	Swerve	$1, 3$	$2, 2$

Choosing a Restaurant

Rebecca

		P	V
Ethan	P	$4, 3$	$1, 1$
	V	$0, 0$	$3, 4$

Demand Bargaining

N players

Each player "demands" a real number in $[0, 10]$

If the demands sum to 10 or less, each player's payoff is her bid

Otherwise players' payoffs are 0

Outline

Strategic Form Games

Solving a Game: Nash Equilibrium

Nash Equilibrium

A strategy profile where no individual has a unilateral incentive to change her behavior

Before we talk about why this is our central solution concept, let's formalize it

Notation

Player i's strategy

- s_i

Set of all possible strategies for Player i

- S_i

Strategy profile (one strategy for each player)

- $\mathbf{s} = (s_1, s_2, \ldots, s_N)$

Strategy profile for all players except i

- $\mathbf{s_{-i}} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$

Different notation for strategy profile

- $\mathbf{s} = (s_i, \mathbf{s_{-i}})$

Selfish Students

		Player 2	
		α	β
Player 1	α	$1, 1$	$3, 0$
	β	$0, 3$	$2, 2$

$S_i = \{\alpha, \beta\}$

4 strategy profiles: $(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta)$

Chicken

		Player 2	
		Straight	Swerve
Player 1	Straight	$0, 0$	$3, 1$
	Swerve	$1, 3$	$2, 2$

$S_i = \{\text{Straight, Swerve}\}$

4 strategy profiles: (Straight, Straight), (Straight, Swerve), (Swerve, Straight), (Swerve, Swerve)

Choosing a Restaurant

		Rebecca	
		P	V
Ethan	P	$4, 3$	$1, 1$
	V	$0, 0$	$3, 4$

$S_E =?$ $S_R =?$

Strategy profiles: ?

Demand bargaining with 3 players

$S_i = [0, 10]$

- Player i can choose any real number between 0 and 10

$\mathbf{s} = (s_1 = 1, s_2 = 4, s_3 = 7) = (1, 4, 7)$

- An example of a strategy profile

$\mathbf{s_{-2}} = (1, 7)$

- Same strategy profile, with player 2's strategy omitted

$\mathbf{s} = (\mathbf{s_{-2}}, s_2) = ((1, 7), 4)$

- Reconstructing the strategy profile

Notating Payoffs

Players' payoffs are defined over strategy profiles

- A strategy profile implies an outcome of the game

Player i's payoff from the strategy profile $\mathbf{s}$ is

$$u_i(\mathbf{s})$$

Player i's payoff if she chooses s_i and others play as in $\mathbf{s_{-i}}$

$$u_i(s_i, \mathbf{s_{-i}})$$

Nash Equilibrium

Consider a game with N players. A strategy profile $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a **Nash equilibrium** of the game if, for every player i

$$u_i(s_i^*, \mathbf{s_{-i}}^*) \geq u_i(s_i', \mathbf{s_{-i}}^*)$$

for all $s_i' \in S_i$

Best Responses

A strategy, s_i, is a **best response** by Player i to a profile of strategies for all other players, $\mathbf{s_{-i}}$, if

$$u_i(s_i, \mathbf{s_{-i}}) \geq u_i(s_i', \mathbf{s_{-i}})$$

for all $s_i' \in S_i$

Best Response Correspondence

Player i's **best response correspondence**, BR_i, is a mapping from strategies for all players other than i into subsets of S_i satisfying the following condition:

- For each $\mathbf{s_{-i}}$, the mapping yields a set of strategies for Player i, $\mathrm{BR}_i(\mathbf{s_{-i}})$, such that s_i is in $\mathrm{BR}_i(\mathbf{s_{-i}})$ if and only if s_i is a best response to $\mathbf{s_{-i}}$

An Equivalent Definition of NE

Consider a game with N players. A strategy profile $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a **Nash equilibrium** of the game if s_i^* is a best response to $\mathbf{s_{-i}}^*$ for each $i = 1, 2, \ldots, N$

Selfish vs. Nice

Nice

		α	β
Selfish	α	$1, 2$	$3, 0$
	β	$0, 1$	$2, 3$

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1^{\checkmark}, 2$	$3, 0$
	β	$0, 1$	$2, 3$

Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1^{\checkmark}, 2$	$3^{\checkmark}, 0$
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Selfish vs. Nice

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Selfish	α	$1^{\checkmark}, 2^{\checkmark}$	$3^{\checkmark}, 0$
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Selfish vs. Nice

		Nice	
		α	β
Selfish	α	$1^{\checkmark}, 2^{\checkmark}$	$3^{\checkmark}, 0$
	β	$0, 1$	$2, 3^{\checkmark}$

Selfish vs. Nice

Nice

Selfish		α	β
	α	$1^{\checkmark}, 2^{\checkmark}$	$3^{\checkmark}, 0$
	β	$0, 1$	$2, 3^{\checkmark}$

Chicken

		Player 2	
		Straight	Swerve
Player 1	Straight	$0, 0$	$3, 1$
	Swerve	$1, 3$	$2, 2$

Chicken

		Player 2	
		Straight	Swerve
Player 1	Straight	$0, 0$	$3, 1$
	Swerve	$1^{\checkmark}, 3$	$2, 2$

Chicken

Player 2

			Swerve
Player 1	Straight	$0, 0$	$3^{\checkmark}, 1$
	Swerve	$1^{\checkmark}, 3$	$2, 2$

Chicken

		Player 2	
		Straight	Swerve
Player 1	Straight	$0, 0$	$3^{\checkmark}, 1^{\checkmark}$
	Swerve	$1^{\checkmark}, 3$	$2, 2$

Chicken

Player 2

Player 1

	Straight	Swerve
Straight	$0, 0$	$3^{\checkmark}, 1^{\checkmark}$
Swerve	$1^{\checkmark}, 3^{\checkmark}$	$2, 2$

Chicken

		Player 2	
		Straight	Swerve
Player 1	Straight	$0, 0$	$3^{\checkmark}, 1^{\checkmark}$
	Swerve	$1^{\checkmark}, 3^{\checkmark}$	$2, 2$

You Solve Choosing a Restaurant

Rebecca

		P	V
Ethan	P	$4, 3$	$1, 1$
	V	$0, 0$	$3, 4$

Another Practice Game

		Player 2	
		L	R
Player 1	U	$10, 2$	$3, 4$
	D	$-1, 0$	$5, 7$

The War of Attrition

2 countries (1 and 2) are fighting over a territory

Each country i decides how long it is willing to hold out, $t_i \geq 0$

The winner is the country that is willing to hold out for the longest time

- If both hold out the same amount of time, they split the territory

The war ends as soon as one country gives in

Country i's Payoffs

Value of winning whole territory is $v_i > 0$

Value of winning half the territory is $\frac{v_i}{2}$

Cost of holding out for length of time t_i is t_i

$$u_1(t_1, t_2) = \begin{cases} -t_1 & \text{if } t_1 < t_2 \\ \frac{v_1}{2} - t_1 & \text{if } t_1 = t_2 \\ v_1 - t_2 & \text{if } t_1 > t_2 \end{cases}$$

Country 1's Best Response if $t_2 < v_1$

If Country 1 chooses $t_1 < t_2$, its payoff is $-t_1$

- Maximized at 0

If Country 1 chooses $t_1 = t_2$, its payoff is $\frac{v_1}{2} - t_1$

If Country 1 chooses $t_1 > t_2$, its payoff is $v_1 - t_2$

Country 1's Best Response if $t_2 < v_1$

If Country 1 chooses $t_1 < t_2$, its payoff is $-t_1$

- Maximized at 0

If Country 1 chooses $t_1 = t_2$, its payoff is $\frac{v_1}{2} - t_1$

If Country 1 chooses $t_1 > t_2$, its payoff is $v_1 - t_2$

Any $t_1 > t_2$ is a best response

Country 1's Best Response if $t_2 = v_1$

If Country 1 chooses $t_1 < t_2$, its payoff is $-t_1$

- Maximized at 0

If Country 1 chooses $t_1 = t_2 = v_1$, its payoff is $\frac{v_1}{2} - t_1 < 0$

If Country 1 chooses $t_1 > t_2$, its payoff is $v_1 - t_2 = 0$

Country 1's Best Response if $t_2 = v_1$

If Country 1 chooses $t_1 < t_2$, its payoff is $-t_1$

- Maximized at 0

If Country 1 chooses $t_1 = t_2 = v_1$, its payoff is $\frac{v_1}{2} - t_1 < 0$

If Country 1 chooses $t_1 > t_2$, its payoff is $v_1 - t_2 = 0$

$t_1 = 0$ or any $t_1 > t_2$ are best responses

Country 1's Best Response if $t_2 > v_1$

If Country 1 chooses $t_1 < t_2$, its payoff is $-t_1$

- Maximized at 0

If Country 1 chooses $t_1 = t_2$, its payoff is $\frac{v_1}{2} - t_1 < 0$

If Country 1 chooses $t_1 > t_2$, its payoff is $v_1 - t_2 < 0$

Country 1's Best Response if $t_2 > v_1$

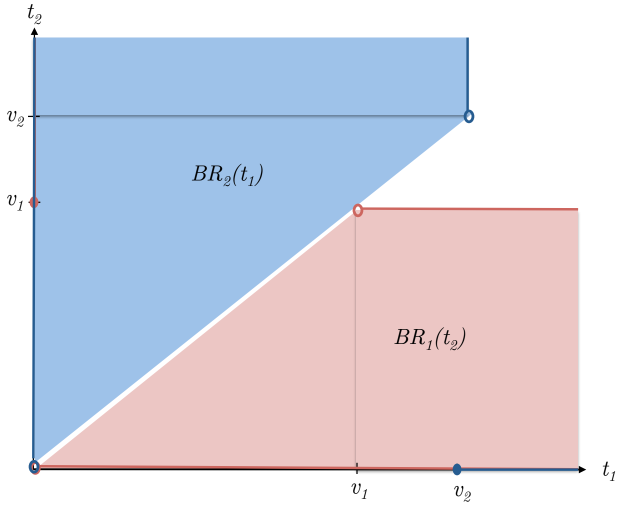
If Country 1 chooses $t_1 < t_2$, its payoff is $-t_1$

- Maximized at 0

If Country 1 chooses $t_1 = t_2$, its payoff is $\frac{v_1}{2} - t_1 < 0$

If Country 1 chooses $t_1 > t_2$, its payoff is $v_1 - t_2 < 0$

$t_1 = 0$ is the best response


t_2
v_2
v_1
$BR_2(t_1)$
$BR_1(t_2)$
v_1
v_2
t_1

Nash Equilibria

$t_1 = 0$ and $t_2 > v_1$

$t_1 > v_2$ and $t_2 = 0$

Why Nash Equilibrium?

No regrets

Social learning

Self-enforcing agreements

Analyst humility

Take Aways

A Nash Equilibrium is a strategy profile where each player is best responding to what all other players are doing

You find a NE by calculating each player's best response correspondence and seeing where they intersect

NE is our main *solution concept* for strategic situations