Game Theory 2: Extensive-Form Games and Subgame Perfection

Dynamics in Games

How should we think of strategic interactions that occur in sequence?

Who moves when?

And what can they do at different points in time?

How do people react to different histories?

Modeling Games with Dynamics

Players

Player function

- Who moves when

Terminal histories

- Possible paths through the game

Preferences over terminal histories

Strategies

A strategy is a complete contingent plan

Player i's strategy specifies her action choice at each point at which she could be called on to make a choice

An Example: International Crises

Two countries (A and B) are competing over a piece of land that B occupies

Country A decides whether to make a demand

If Country A makes a demand, B can either acquiesce or fight a war

If A does not make a demand, B keeps land (game ends)

A's best outcome is Demand followed by Acquiesce, worst outcome is Demand and War

B's best outcome is No Demand and worst outcome is Demand and War

An Example: International Crises

A can choose: Demand (D) or No Demand (ND)

B can choose: Fight a war (W) or Acquiesce (A)

Preferences

$$u_A(D, A) = 3 > u_A(ND, A) = u_A(ND, W) = 2 > u_A(D, W) = 1$$

$$u_B(ND, A) = u_B(ND, W) = 3 > u_B(D, A) = 2 > u_B(D, W) = 1$$

How can we represent this scenario as a game (in strategic form)?

International Crisis Game: NE

Country B

		W	A
Country A	D	$1, 1$	$3, 2$
	ND	$2, 3$	$2, 3$

International Crisis Game: NE

		Country B	
		W	A
Country A	D	$1, 1$	$3, 2$
	ND	$2^{\checkmark}, 3$	$2, 3$

International Crisis Game: NE

		Country B	
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Country A	D	$1, 1$	$3^{\checkmark}, 2$
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International Crisis Game: NE

		Country B	
		W	A
Country A	D	$1, 1$	$3^{\checkmark}, 2^{\checkmark}$
	ND	$2^{\checkmark}, 3$	$2, 3$

International Crisis Game: NE

		Country B	
		W	A
Country A	D	$1, 1$	$3^{\checkmark}, 2^{\checkmark}$
	ND	$2^{\checkmark}, 3^{\checkmark}$	$2, 3^{\checkmark}$

International Crisis Game: NE

Country B

Country A		W	A
	D	$1, 1$	$3^{\checkmark}, 2^{\checkmark}$
	ND	$2^{\checkmark}, 3^{\checkmark}$	$2, 3^{\checkmark}$

International Crisis Game: NE

		Country B	
		W	A
Country A	D	$1, 1$	$3^{\checkmark}, 2^{\checkmark}$
	ND	$2^{\checkmark}, 3^{\checkmark}$	$2, 3^{\checkmark}$

- Is there something funny here?

International Crisis Game: NE

Country A		Country B: W	Country B: A
	D	$1, 1$	$3^{\checkmark}, 2^{\checkmark}$
	ND	$2^{\checkmark}, 3^{\checkmark}$	$2, 3^{\checkmark}$

- Is there something funny here?
- Specifically, (ND, W)?

International Crisis Game: NE

		Country B	
		W	A
Country A	D	$1, 1$	$3^{\checkmark}, 2^{\checkmark}$
	ND	$2^{\checkmark}, 3^{\checkmark}$	$2, 3^{\checkmark}$

- Is there something funny here?
- Specifically, (ND, W)?
- The threat of war deters the demand, but would B follow through?

Non-Credible Threats

The equilibrium (ND, W) depends on a "non-credible threat"

Once A makes a demand, B does not want to fight a war

But to rule out such behavior, we need a stronger solution concept

One that incorporates the fact that actions are taken in sequence

Why Rule out Non-credible Threats

Equilibrium as a steady state

War is only a best-response for B because when no demand is made, B is indifferent

If A accidentally made a demand, war is not a sequential best-response for B. B should acquiesce instead

- Read the strategy W as "if A makes a demand, I will go to war"

Subgame Perfect Nash Equilibrium

A strategy specifies what a player will do at every decision point

- Complete contingent plan

Strategy in a SPNE must be a best-response at each node, given the strategies of other players

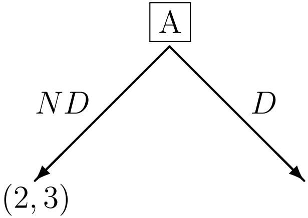
Backward Induction

But First!

Let's introduce a way of incorporating the timing of actions into the game explicitly

Use a game tree to represent the sequential aspect of choices

An Example: International Crises



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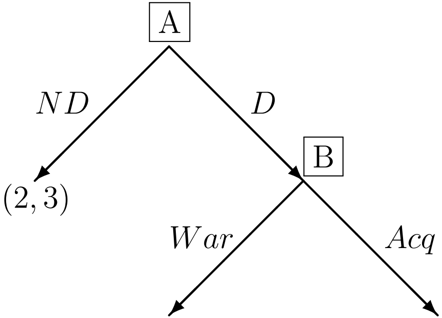

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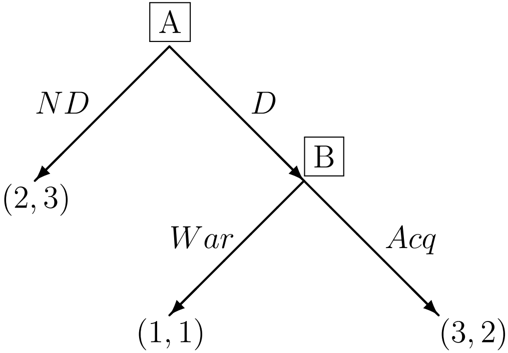

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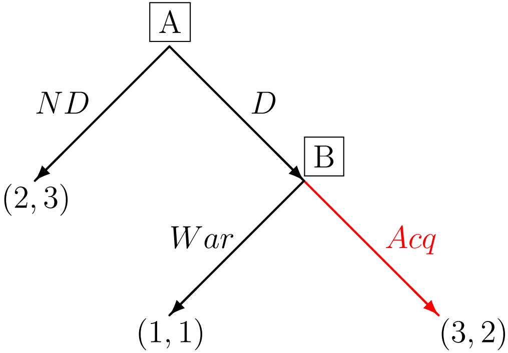

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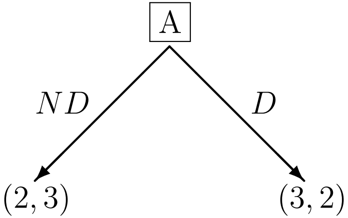

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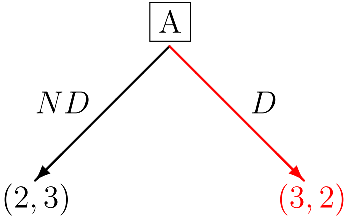

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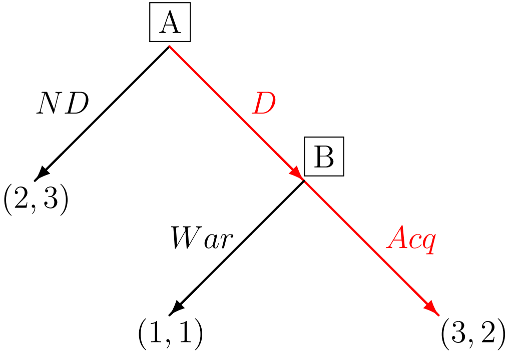


Another Example

		Player 2	
		C	D
Player 1	A	$5, 3$	$0, 0$
	B	$0, 0$	$3, 5$

Another Example

		Player 2	
		C	D
Player 1	A	$5^{\checkmark}, 3^{\checkmark}$	$0, 0$
	B	$0, 0$	$3^{\checkmark}, 5^{\checkmark}$

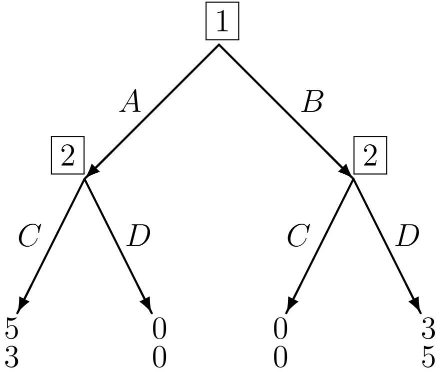
Another Example

Player 2

Player 1		C	D
	A	$5^{\checkmark}, 3^{\checkmark}$	$0, 0$
	B	$0, 0$	$3^{\checkmark}, 5^{\checkmark}$

Suppose that player 1 moves first and player 2 moves second.

Another Example



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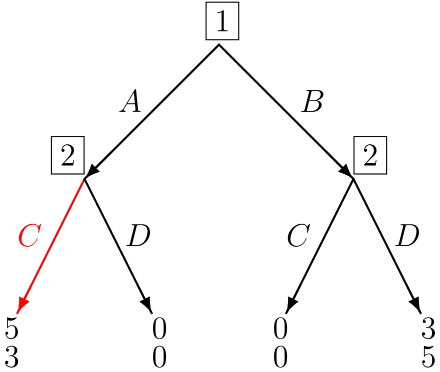

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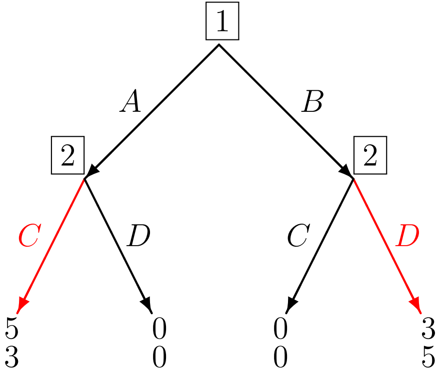

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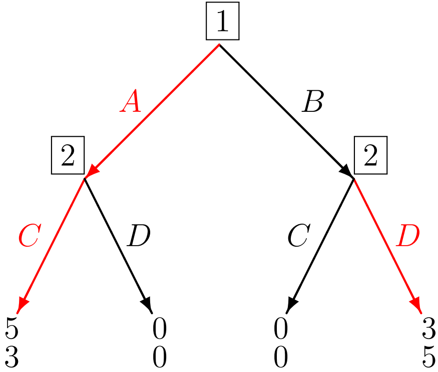

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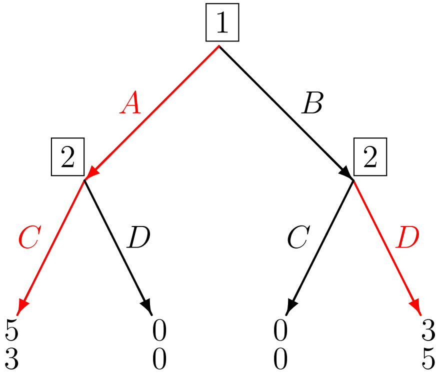
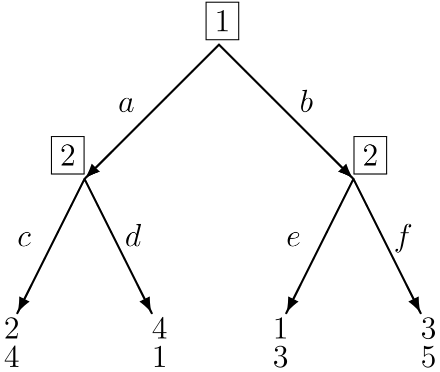


SPNE: $(A, (C, D))$

Practice Game



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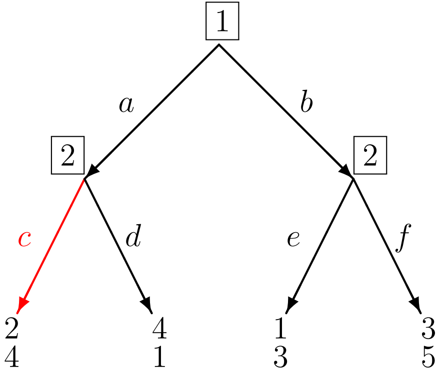

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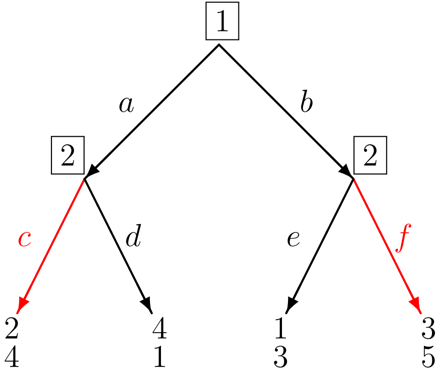

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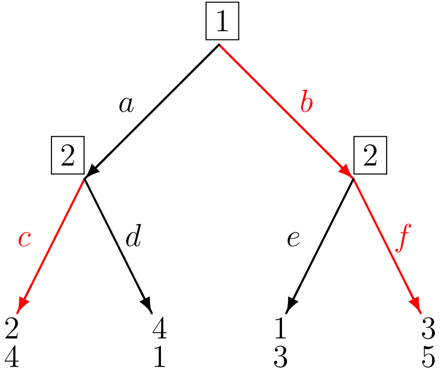

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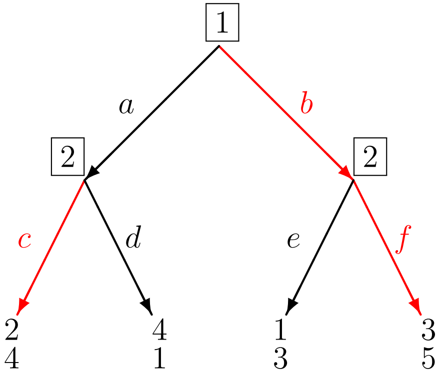
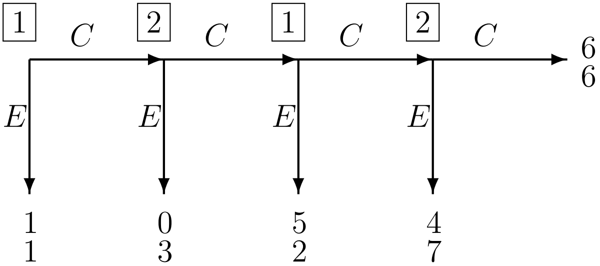


SPNE: $(b, (c, f))$

The Centipede Game



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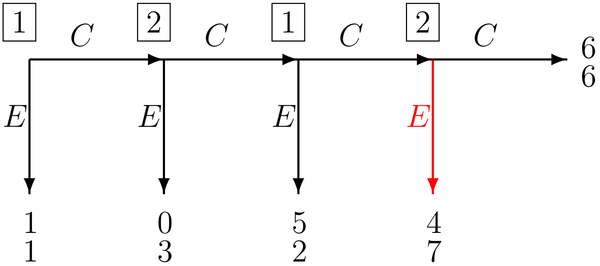

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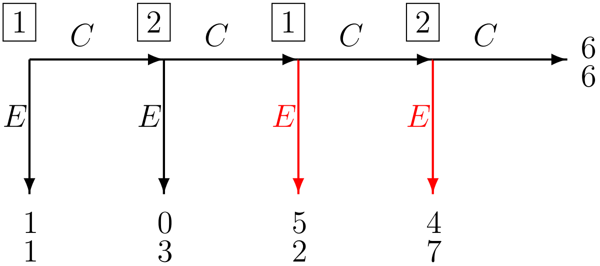

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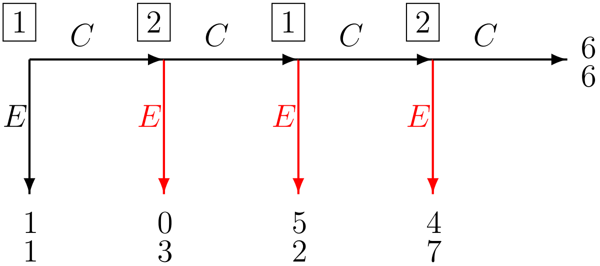

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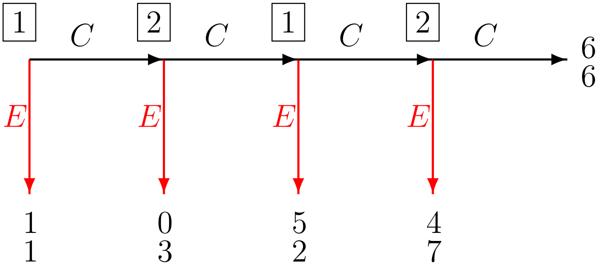

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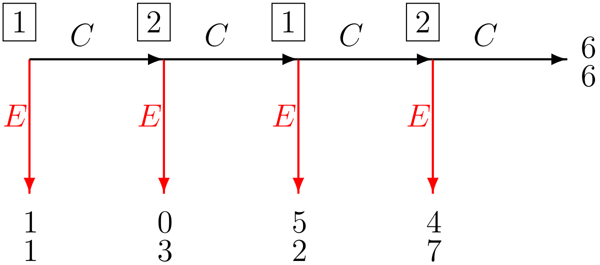
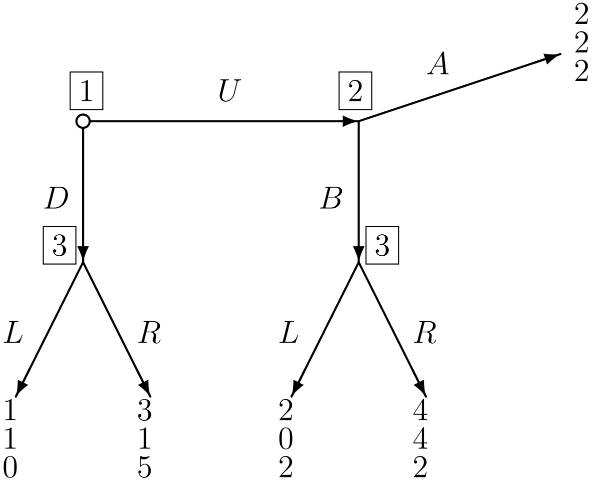


Unique SPNE: $((E, E), (E, E))$

Equilibrium payoffs $(1, 1)$

Pareto dominated by 3 outcomes!

Multiple Equilibria



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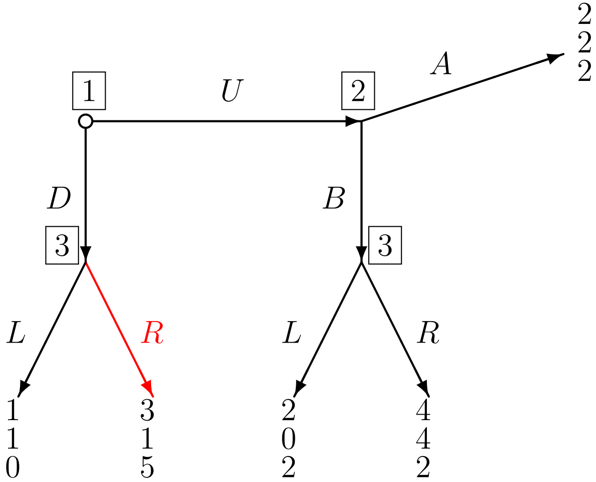

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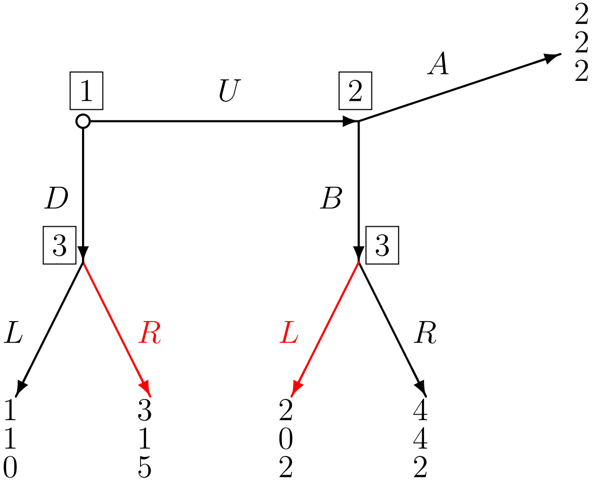

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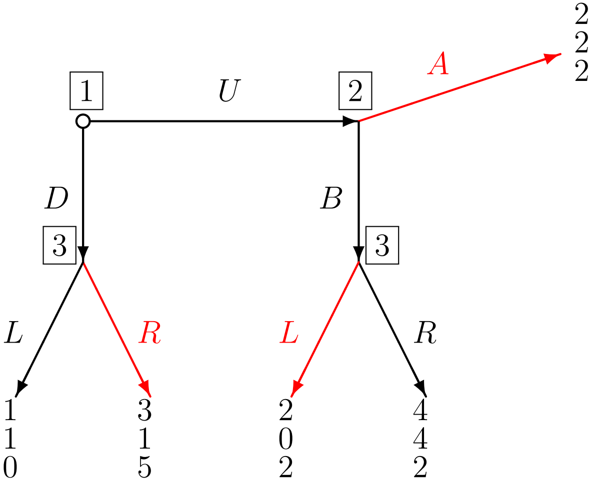

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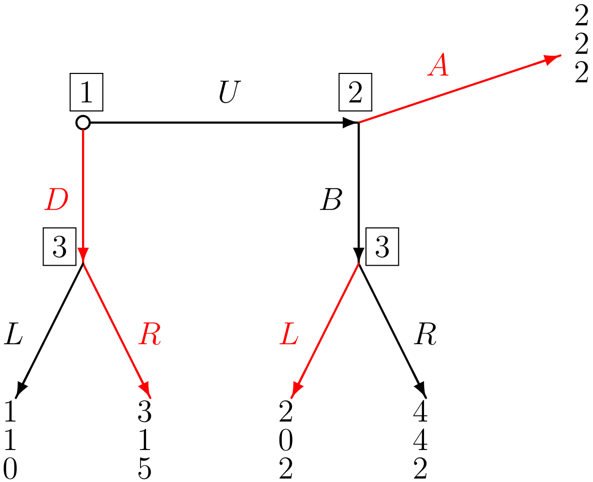
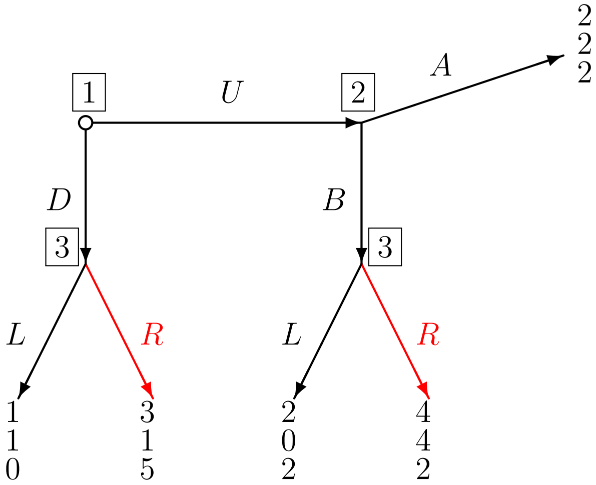


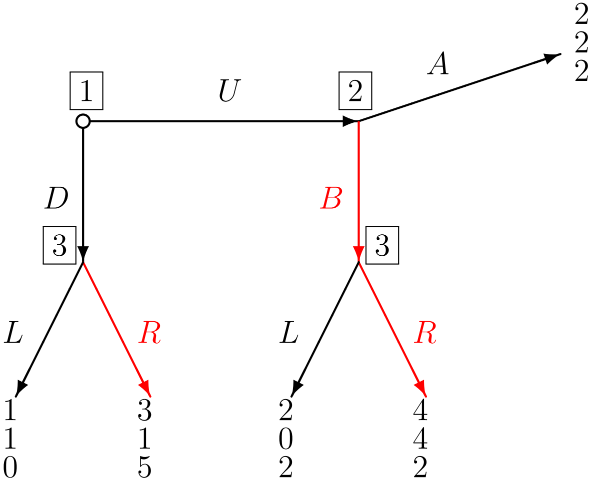
SPNE 1: (D, A, (R,L))

Multiple Equilibria



SPNE 1: (D, A, (R,L))

Multiple Equilibria



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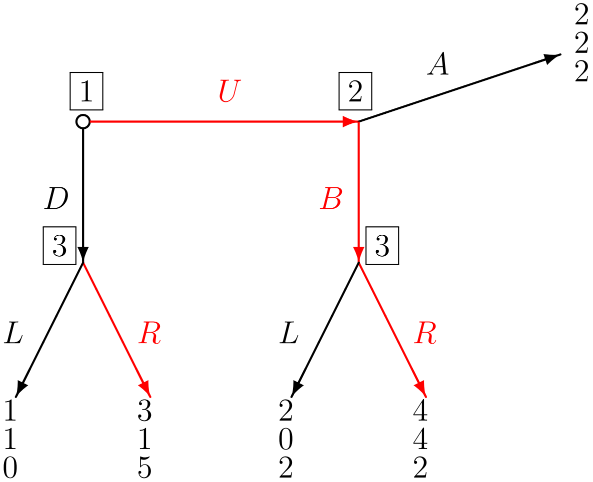

SPNE 1: (D, A, (R,L))
SPNE 2: (U,B,(R,R))

A Familiar Example: Public Good in a Team

Two players: 1 & 2

Each can choose a level to contribute to a public good: s_i

Payoff for individual i are

$$u_i(s_1, s_2) = s_1 + s_2 + \frac{s_1 s_2}{2} - \frac{s_i^2}{2}$$

Nash Equilibrium

$$s_1^* = 2 \qquad s_2^* = 2$$

Individual player's equilibrium payoff:

$$2 + 2 + \frac{2 \cdot 2}{2} - \frac{2^2}{2} = 4$$

Consider an extensive form version

Player 1 must make her choice first

Before Player 2 decides how much to put in, she observes how much Player 1 puts in

How might this change contributions?

We will use backward induction

Best Response for Player 2

The payoff function for player 2:

$$u_2(s_1, s_2) = s_1 + s_2 + \frac{s_1 s_2}{2} - \frac{s_2^2}{2}$$

How do we determine the best response of player 2?

Best Response for Player 2

The payoff function for player 2:

$$u_2(s_1, s_2) = s_1 + s_2 + \frac{s_1 s_2}{2} - \frac{s_2^2}{2}$$

How do we determine the best response of player 2?

$$\frac{\partial u_2(s_1, s_2)}{\partial s_2} = 1 + \frac{s_1}{2} - s_2$$

Best Response for Player 2

The payoff function for player 2:

$$u_2(s_1, s_2) = s_1 + s_2 + \frac{s_1 s_2}{2} - \frac{s_2^2}{2}$$

How do we determine the best response of player 2?

$$\frac{\partial u_2(s_1, s_2)}{\partial s_2} = 1 + \frac{s_1}{2} - s_2$$

Setting equal to zero ($\frac{\partial u_2(s_1,s_2)}{\partial s_1} = 0$), Player 2's best-response to s_1 is

$$\mathrm{BR}_2(s_1) = 1 + \frac{s_1}{2}$$

Best Response for Player 1

Player 1's best response must account for how Player 2 will respond to whatever she chooses

Best Response for Player 1

Player 1's best response must account for how Player 2 will respond to whatever she chooses :

$$u_1(s_1, \mathrm{BR}_2(s_1))$$

Best Response for Player 1

Player 1's best response must account for how Player 2 will respond to whatever she chooses :

$$u_1(s_1, \mathrm{BR}_2(s_1))$$

$$u_1(s_1, \mathrm{BR}_2(s_1)) = s_1 + \mathrm{BR}_2(s_1) + \frac{s_1 \times \mathrm{BR}_2(s_1)}{2} - \frac{s_1^2}{2}$$

Best Response for Player 1

Player 1's best response must account for how Player 2 will respond to whatever she chooses :

$$u_1(s_1, \mathrm{BR}_2(s_1))$$

$$u_1(s_1, \mathrm{BR}_2(s_1)) = s_1 + \mathrm{BR}_2(s_1) + \frac{s_1 \times \mathrm{BR}_2(s_1)}{2} - \frac{s_1^2}{2}$$

$$u_1(s_1, \mathrm{BR}_2(s_1)) = s_1 + \left(1 + \frac{s_1}{2}\right) + \frac{s_1\left(1 + \frac{s_1}{2}\right)}{2} - \frac{s_1^2}{2}$$

Best Response for Player 1

Player 1's best response must account for how Player 2 will respond to whatever she chooses :

$$u_1(s_1, \mathrm{BR}_2(s_1))$$

$$u_1(s_1, \mathrm{BR}_2(s_1)) = s_1 + \mathrm{BR}_2(s_1) + \frac{s_1 \times \mathrm{BR}_2(s_1)}{2} - \frac{s_1^2}{2}$$

$$u_1(s_1, \mathrm{BR}_2(s_1)) = s_1 + \left(1 + \frac{s_1}{2}\right) + \frac{s_1\left(1 + \frac{s_1}{2}\right)}{2} - \frac{s_1^2}{2}$$

$$u_1(s_1, \mathrm{BR}_2(s_1)) = 1 + \frac{3}{2}s_1 + \frac{s_1}{2} + \frac{s_1^2}{4} - \frac{s_1^2}{2}$$

Best Response for Player 1

We can write Player 1's problem as:

$$u_1(s_1, \mathrm{BR}_2(s_1)) = 1 + 2s_1 - \frac{s_1^2}{4}$$

Solve for Player 1's optimal choice:

$$2 - \frac{s_1^*}{2} = 0$$

$$s_1^* = 4$$

Best Response for Player 1

We can write Player 1's problem as:

$$u_1(s_1, \mathrm{BR}_2(s_1)) = 1 + 2s_1 - \frac{s_1^2}{4}$$

Solve for Player 1's optimal choice:

$$2 - \frac{s_1^*}{2} = 0$$

$$s_1^* = 4$$

Go back to Player 2:

$$s_2^* = \mathrm{BR}_2(s_1^*) = \mathrm{BR}_2(4) = 1 + \frac{4}{2} = 3$$

Public Good in a Team

So each player contributes more:

$$s_1^* = 4 \qquad s_2^* = \mathrm{BR}_2(s_1^*) = 3$$

Public Good in a Team

So each player contributes more:

$$s_1^* = 4 \qquad s_2^* = \mathrm{BR}_2(s_1^*) = 3$$

and equilibrium utilities:

$$u_1^* = 5 \qquad u_2^* = 8.5$$

They each are better off, but it's better to move second

Subgame Perfect Nash Equilibrium

Subgame Perfect Nash Equilibrium is a refinement of Nash Equilibrium

It rules out equilibria that rely on incredible threats in a dynamic environment

All SPNE are identified by backward induction