FISCAL POLICY AND DEBT MANAGEMENT WITH INCOMPLETE MARKETS*

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A Ramsey planner chooses a distorting tax on labor and manages a portfolio of securities in an economy with incomplete markets. We develop a method that uses second order approximations of Ramsey policies to obtain formulas for conditional and unconditional moments of government debt and taxes that include means and variances of the invariant distribution as well as speeds of mean reversion. The asymptotic mean of the planner's portfolio minimizes a measure of fiscal risk. We obtain analytic expressions that approximate moments of the invariant distribution and apply them to data on a primary government deficit, aggregate consumption, and returns on traded securities. For U.S. data, we find that the optimal target debt level is negative but close to zero, the invariant distribution of debt is very dispersed, and mean reversion is slow. *JEL* Codes: E62, H63, G18.

I. Introduction

This article models a Ramsey planner who optimally manages a portfolio of debts and other securities to smooth fluctuations in tax distortions in an incomplete markets economy subject to aggregate shocks. Within a production economy without capital, the government raises revenue by issuing securities and imposing a linear tax on labor income, which it spends on exogenous government expenditures, payouts on government securities, and transfers. The government and private agents trade an exogenously specified set of risky securities whose returns depend on the aggregate state. An economy with complete markets and an economy with a one-period risk-free bond only are interesting special cases.

We make extensive use of an approximation to a Ramsey plan that we construct from second-order perturbations around current levels of government debt. We confirm that these quadratic

*We thank Fernando Alvarez, David Backus, V.V. Chari, Maryam Farboodi, Emmanuel Farhi, Xavier Gabaix, Lars Peter Hansen, Nicola Pavoni, Ali Shourideh, Harald Uhlig, Pierre Yared, Stephen Zeldes, and seminar participants at numerous conferences and seminars for helpful comments.

The Quarterly Journal of Economics (2017), 617–663. doi:10.1093/qje/qjw041.
Advance Access publication on November 15, 2016.

approximations are accurate by comparing them to solutions obtained using numerical methods. Under conditions that we describe, the approximating laws of motion are linear functions of the aggregate shocks and the current level of government debt. Our quadratic approximations then enable analytic and interpretable expressions for means, variances, and rates of convergence to an invariant distribution of debt, tax revenues, and tax rates.[1] Empirical counterparts to our expressions for these objects can be constructed from data on the primary government deficit, aggregate consumption, and returns on securities traded by the government. We show that the mean of the government's optimal debt portfolio eventually minimizes a particular criterion that measures fiscal risk.

To isolate underlying principles, we start with a baseline setting in which agents have quasilinear preferences and the market structure is restricted to a single security whose payout we allow to be correlated with the government purchase process. The joint distribution of returns and government purchases is independent and i.i.d. over time. From the planner's Euler equations, we establish the existence of an invariant distribution of government debt. Up to third-order terms, we show that the drift in the dynamics of debt is proportional to the covariance of returns with total government spending (debt service plus exogenous government purchases). A level of debt that minimizes the variance of total government spending sets this covariance to zero and serves as a point of attraction for the stochastic process for debt. The speed of mean reversion is inversely proportional to the variance of the return on the security, and the variance of the invariant distribution is proportional to the amount of risk that the government bears at its risk-minimizing debt level. Later sections of the article show that the principle that government debt approaches a level that minimizes fiscal risk extends well beyond our baseline case.

Allowing trade in more securities yields additional insights. If returns satisfy a spanning condition, the planner can replicate a complete markets allocation like Lucas and Stokey's (1983). When that spanning condition is not satisfied, being able to trade more securities decreases the speed of convergence to the invariant distribution because additional securities facilitate hedging and

1. We can also use our quadratic approximation to get analytic expressions for other moments.

thereby lower the cost of being away from a long-run target level of government debt. By assuming two particular securities, a consol and a short-term security, we derive prescriptions for optimal maturity management. In this two-security case, the riskiness of the return on the short-maturity asset relative to that on the consol affects the average maturity of the total debt. In particular, if the return on the long maturity bond is riskier than the return on the short-maturity bill, then the optimal maturity of the planner's portfolio is inversely proportional to total public debt and most adjustment to aggregate shocks is done with the bill. We extend the analysis to incorporate risk aversion and more general shock processes. We show that insights from the baseline model apply provided that we use concepts of "effective returns" and "effective shocks" — returns on the government debt portfolio and innovations to the present discounted value of the primary government deficit adjusted by marginal utilities of consumption, respectively.

In a quantitative section, we pursue two goals: (i) to verify the accuracy of our quadratic approximations using global numerical methods; and (ii) to study implications of the model for realistic shock and return processes. To this end, we use U.S. data to calibrate plausible shock and return processes. Our analytical expressions are shown to be accurate in the calibrated model. We find that asymptotically the optimal level of government debt is close to zero and that the optimal policy for government debt displays slow mean reversion (a half-life of almost 250 years). These results are driven by the fact that a significant amount of variation in returns to the U.S. portfolio is uncorrelated with output; that implies that holding large quantities of debt or assets would frustrate hedging objectives.

To focus on some important forces, our article obviously shuts down forces emphasized in other theories of optimal levels of government debt. For example, by allowing a government each period to choose whether to service its debt, the literature on sovereign debt focuses attention on how the adverse consequences of default endogenously generate incentives to repay debt obligations. The government in our model has no default option and requires no incentives to repay. This eliminates the design of incentives to induce payment as determinants of the level of government debt and its maturity composition and puts the hedging considerations on which we focus front and center. Our model describes optimal fiscal policy of a government that never contemplates dishonoring its debts. (We like to think of the U.S. and some European governments as being in this situation.) Additionally, we focus on real debt. Extending our approach to economies with possibilities of default and monetary economies is straightforward but space-consuming as it would require us to introduce several layers of additional complications to our model. We leave that for future work.

I.A. *Relationships to Literatures*

Our article builds on a large literature about a Ramsey planner who chooses a competitive equilibrium with distorting taxes once and for all at time 0.[2] Many of these papers assume either complete markets as in Lucas and Stokey (1983), Buera and Nicolini (2004), Angeletos (2002), or a one-period risk-free bond only and quasilinear preferences as in Barro (1979) and Aiyagari et al. (2002). In contrast, our analysis allows a more general incomplete markets structure and risk aversion. In both complete market economies and quasilinear settings with a risk-free bond only, *any* level of debt is optimal in the sense that the Ramsey planner sets a time 0 conditional mathematical expectation of public debt in all future periods equal to initial debt. We show that this result is fragile: small departures from the assumptions in those earlier papers imply that, driven by hedging considerations, starting from any initial debt, government debt converges to a unique risk-minimizing level.

In a related context, Barro (1999, 2003) studies tax smoothing in an environment in which revenue needs are deterministic but refinancing opportunities are stochastic. In Barro's setting, it is optimal for a government to issue a consol as a way to insulate intertemporal tax smoothing motives from concerns about rolling over short maturity debt at uncertain prices. In contrast, our analysis allows both revenue needs and returns on the debt to be stochastic. We estimate empirically relevant properties of returns on debt and then find an optimal government portfolio associated with those returns.

Technically, our article is closely related to Aiyagari et al. (2002). Those authors include an analysis of an economy in which a representative agent has quasilinear preferences. In addition to a linear labor tax, they allow a uniform nonnegative lump-sum transfer. They find that there is a continuum of invariant

2. For instance, see Lucas and Stokey (1983), Chari et al. (1994), Aiyagari et al. (2002), Farhi (2010).

distributions for debt, all of which feature a zero labor tax rate and debt levels that are negative and sufficiently large in absolute value to finance all government expenditures from the government's interest revenues, with nonnegative transfers absorbing all aggregate fluctuations by adjusting one to one with the aggregate shock.

In Section III.A, we depart from Aiyagari et al. (2002) and model optimal transfers as arising from an explicit redistribution motive by including agents who cannot afford to pay positive lump sum taxes. We show that as long as the utility functions of those agents are strictly concave and the planner cares about them, the Ramsey plan ultimately targets a (generally unique) level of debt that minimizes risk as in our representative agent settings. The invariant distribution studied by Aiyagari et al. (2002) emerges only in a limit as the risk aversion of all recipients of transfers goes to zero.

The equilibrium approximation tools that we apply in this article are complementary to ones used by Faraglia, Marcet, and Scott (2012), Lustig, Sleet, and Yeltekin (2008), and Siu (2004), who numerically study optimal Ramsey plans in specific incomplete markets settings. Our approximation method allows us to derive closed-form expressions for the invariant distribution of debt and taxes that illuminate underlying forces. Our work is also related to Debortoli, Nunes, and Yared (2016) who numerically characterize optimal debt management when a government cannot commit to future taxes.

Our theory of government portfolio management shares features of the single-investor optimal portfolio theory of Markowitz (1952) and Merton (1969). Bohn (1990) and Lucas and Zeldes (2009) use insights from the single-investor literature to study portfolio choices of a government in partial equilibrium settings after having specified a government loss function. We also build on the work of Farhi (2010) who derives the CCAPM equations in the incomplete market Ramsey settings similar to ours. We show that the Ramsey planner chooses a portfolio to minimize a measure of fiscal risk and derive closed-form expressions for the optimal portfolio.

The remainder of this article is organized as follows. In Section II, we analyze a streamlined setting in which only one risky security can be traded and the representative agent has quasilinear preferences. In Section III, we extend the analysis to include multiple assets, persistent shocks, concerns for redistribution, and risk aversion. In Section IV, we study a quantitative example with parameters calibrated to U.S. data.

II. Quasilinear Preferences

We begin with a streamlined setting. Time is discrete and infinite with periods denoted $t = 0, 1, \ldots$ Each of a measure 1 of identical agents has preferences over consumption and labor supply sequences $\{c_t, l_t\}_t$ that are ordered by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(c_t - \frac{1}{1+\gamma} l_t^{1+\gamma} \right), \tag{1}$$

where $\mathbb{E}_t$ is a mathematical expectations operator conditioned on time t information and $\beta \in (0, 1)$ is a time discount factor. One unit of labor produces one unit of a nonstorable single good that can be consumed by households or the government. Feasibility requires

$$c_t + g_t = l_t, \quad t \geq 0, \tag{2}$$

where g_t denotes government consumption.

The government imposes a flat tax at rate τ_t on labor earnings and buys or sells a single one-period security having an exogenous state-contingent payoff p_t. Consumers sell or buy that same security, so it is in zero net supply each period. Let B_t be the number of securities that the government sells in period t at price q_t. Government budget constraints are

$$g_t + p_t \mathrm{B}_{t-1} = \tau_t l_t + q_t \mathrm{B}_t, \quad t \geq 0. \tag{3}$$

A probability measure $\pi(ds)$ over a compact set S governs an exogenous i.i.d. shock s_t that determines both government purchases and payoffs on the single security, positive random variables g, p with means $\bar{g}$, $\bar{p}$.

We let $s^t = (s_0, \ldots, s_t)$ denote a history of shocks. We often use x_t to denote a random variable x with a time t conditional distribution that is a function of history s^{t-1}. It is convenient to define $B_t \equiv q_t \mathrm{B}_t$ and $R_{t+1} \equiv \frac{p_{t+1}}{q_t}$ and to rewrite the government's time t budget constraint (3) as

$$g_t + R_t B_{t-1} = \tau_t l_t + B_t.$$

A representative agent's time t budget constraint is

$$c_t + b_t = (1 - \tau_t) l_t + R_t b_{t-1}, \tag{4}$$

where b_t is his purchase of the single security. The period t market clearing condition for the security is

$$b_t = B_t. \tag{5}$$

We exogenously confine government debt to a compact set

$$B_t \in \left[\underline{B}, \overline{B}\right]. \tag{6}$$

The assumption of compactness of the feasible debt simplifies the analysis. We make the bounds sufficiently large that they do not affect the properties of the joint invariant distributions of government debt and the tax rate that we analyze below.

DEFINITION 1. A competitive equilibrium given an initial government debt B_{-1} at $t = 0$ is a sequence $\{c_t, l_t, B_t, b_t, R_t, \tau_t\}_t$ such that (i) $\{c_t, l_t, b_t\}_t$ maximize equation (1) subject to the budget constraints (4); and (ii) constraints (2), (3), (5), and (6) are satisfied. An optimal competitive equilibrium given B_{-1} is a competitive equilibrium that has the highest value of equation (1).

The single-security incomplete markets models of Barro (1979) and Aiyagari et al. (2002) assume that the security's payout is risk-free, a special case of our setup in which $p(s)$ is independent of s. The payoff shocks aim to capture a general setting where macroeconomic shocks such as expenditure or productivity affect returns on the government's portfolio either directly or indirectly through the response of tax policies to these shocks. We disentangle these concerns by first studying a quasilinear economy where returns, modeled directly using $p(s)$ on traded securities, are arbitrarily correlated with macroeconomic shocks, each other and across time. In the baseline outlined above we have correlated shocks that drive expenditures and returns but are i.i.d. across time. In Sections III.B and III.C we allow for multiple assets that can be long-lived and in positive net supply (e.g., Lucas trees) and also more general shocks that follow Markov processes. Furthermore, in Section III.D we show that an economy with risk-averse agents who trade a riskless bond closely resembles our quasi-linear setup where payoffs are risky and chosen to be positively correlated with expenditure shocks.

The representative consumer's first-order necessary conditions for an optimum imply that

$$1 - \tau_t = l_t^{\gamma},\ \mathbb{E}_{t-1} R_t = \frac{1}{\beta}. \tag{7}$$

The security price q_t satisfies $q_t = \beta \bar{p}$, so the return on the security $R_t(s^t) = \frac{p(s_t)}{\beta \bar{p}}$. Substitute equation (7) into the consumer's budget constraint to obtain

$$c_t = l_t^{1+\gamma} + R_t B_{t-1} - B_t. \tag{8}$$

Use equation (8) to eliminate c_t from the feasibility condition (2) to obtain the following implementability constraints:

$$l_t - l_t^{1+\gamma} + B_t = R_t B_{t-1} + g_t. \tag{9}$$

LEMMA 1. If $\{c_t,\ l_t,\ B_t,\ b_t,\ R_t,\ \tau_t\}_t$ is a competitive equilibrium given B_{-1} then $\{l_t, B_{t-1}\}_t$ satisfies (6) and (9) for all $t \geq 0$. If $\{l_t, B_{t-1}\}_t$ satisfies (6) and (9) for given B_{-1} and all $t \geq 0$ then there exist $\{c_t, b_t, R_t, \tau_t\}_t$ such that $\{c_t, l_t, B_t, b_t, R_t, \tau_t\}_t$ is a competitive equilibrium given B_{-1}.

Lemma 1 allows us to compute an optimal competitive equilibrium allocation and a government debt process by solving

$$\max_{\{l_t, B_t\}_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(R_t B_{t-1} - B_t) + \frac{\gamma}{1+\gamma} l_t^{1+\gamma} \right], \tag{10}$$

where maximization is subject to constraints (6) and (9). The objective function in equation (10) is a version of equation (1) in which we have used equation (8) to eliminate c_t.

Online Appendix I.A shows that it is optimal to set the tax rate to the left of the peak of the Laffer curve, which implies that the optimal tax rate τ_t and labor supply l_t are described by one-to-one mappings from total tax revenues $Z_t = \tau_t l_t$. Tax revenues are bounded from above by the level $\overline{Z}$ associated with a tax rate at the peak of the Laffer curve.[3] For a given level of tax revenues Z, the corresponding tax rate $\tau(Z)$ and labor supply $l(Z)$ satisfy

$$\begin{aligned} Z &= \tau(Z)(1-\tau(Z))^{\frac{1}{\gamma}} \\ &= l(Z) - l(Z)^{1+\gamma}, \end{aligned} \tag{11}$$

which are well defined for all $Z \leq \overline{Z}$. Functions $l(\cdot)$, $-\tau(\cdot)$ are decreasing. Let $\Psi(Z) \equiv \frac{1}{1+\gamma} l(Z)^{1+\gamma}$ be the utility cost of supplying labor required to raise tax revenues Z. Ψ is strictly decreasing, strictly concave, differentiable on $(-\infty, \overline{Z}]$, and satisfies Inada conditions $\lim_{Z\to-\infty}\Psi'(Z) = 0$ and $\lim_{Z\to\overline{Z}}\Psi'(Z) = -\infty$.

An optimal value function $V(B_-)$ for problem (10) satisfies the Bellman equation

(12)
$$V(B_-) = \max_{Z(\cdot),B(\cdot)} \int \left[(R(s)B_- - B(s)) + \gamma\Psi(Z(s)) + \beta V(B(s))\right]\pi(ds),$$

where maximization is subject to $Z(s) \leq \overline{Z}$, $B(s) \in [\underline{B}, \overline{B}]$, and

$$Z(s) + B(s) = R(s)B_- + g(s) \quad \text{for all } s. \tag{13}$$

Strict concavity and differentiability of Ψ implies that V is also strictly concave and differentiable. Policy functions $\tilde{B}(s, B_-)$ and $\tilde{Z}(s, B_-)$ attain the right side of Bellman equation (12). Let $\tilde{\tau}(s, B_-)$ denote the associated optimal tax rate policy. Gross government expenditures $E(s, B_-)$, an important endogenous variable, are

$$E(s, B_-) = R(s)B_- + g(s), \tag{14}$$

which equals government expenditures including interest and repayment of government debt. Aggregate shocks have effects on $E(s, B_-)$ that depend partly on government debt B_-.

We begin our analysis by stating a lemma that summarizes some key properties of optimal policy rules.

LEMMA 2. $\tilde{B}$, $\tilde{Z}$, and $\tilde{\tau}$ are increasing in E in the sense that $E(s'', B_-'') > E(s', B_-')$ implies $\tilde{B}(s'', B_-'') \geq \tilde{B}(s', B_-')$, $\tilde{Z}(s'', B_-'') \geq \tilde{Z}(s', B_-')$, and $\tilde{\tau}(s'', B_-'') \geq \tilde{\tau}(s', B_-')$ with strict inequalities if $\tilde{B}(s'', B_-''), \tilde{B}(s', B_-') \in (\underline{B}, \overline{B})$.

3. The expression for the maximum revenue is $\overline{Z} = \gamma\left(\frac{1}{1+\gamma}\right)^{1+\frac{1}{\gamma}}$.

Let $\{\tilde{B}_t, \tilde{Z}_t\}_t$ be the optimum process generated by policy functions $\tilde{B}(s, B_{-})$ and $\tilde{Z}(s, B_{-})$. First-order conditions associated with the maximization problem (12) imply that if $\tilde{B}_t$ is interior, then the marginal social value of assets $V'(\tilde{B}_t)$ satisfies[4]

(15)
$$V'(\tilde{B}_t) = \beta \mathbb{E}_t R_{t+1} V'(\tilde{B}_{t+1}) = \mathbb{E}_t V'(\tilde{B}_{t+1}) + \beta \mathrm{cov}_t(R_{t+1}, V'(\tilde{B}_{t+1})).$$

Monotonicity of the policy functions asserted in Lemma 2 together with (15) allow us to prove:

PROPOSITION 1. The optimal process $\{\tilde{B}_t, \tilde{Z}_t\}_t$ has a unique invariant distribution.

To enable us to characterize this invariant distribution, a key concept will be the level of debt

$$B^* \equiv \arg\min_B \mathrm{var}\big(RB + g\big) = -\frac{\mathrm{cov}\,(R, g)}{\mathrm{var}\,(R)}. \tag{16}$$

We assume that probability measure π is such that $B^* \in (\underline{B}, \overline{B})$ and that $\overline{B}$ is weakly below the natural debt limit. We call B^* the risk-minimizing level of debt. Let $Z^* \equiv \bar{g} + \frac{1-\beta}{\beta} B^*$ be the constant tax revenues that satisfy the government's budget constraint on average if $B_t = B^*$ for all t.

II.A. Perfectly Correlated Shocks: The Exact Characterization

We first consider a special case in which p and g are perfectly correlated that illustrates key economic forces that determine the long-run behavior of debt and taxes more generally.

PROPOSITION 2. Suppose that p and g are perfectly correlated. Then $\tilde{B}_t \to B^*$, $\tilde{Z}_t \to Z^*$ a.s.

Proof. If p and g are perfectly correlated then $\mathrm{cov}(E(\cdot, B), R(\cdot)) \geq 0$, if $B \geq B^*$, $\mathrm{cov}(E(\cdot, B), R(\cdot)) \leq 0$ if $B \leq B^*$, and $E(s, B)$ is independent of s if and only if $B = B^*$. The monotonicity of policy functions established in Lemma 2 and concavity of V imply that $\mathrm{cov}_t(R_{t+1}, V'(\tilde{B}_{t+1})) \leq 0$ if $\tilde{B}_t \geq B^*$ and $\mathrm{cov}_t(R_{t+1}, V'(\tilde{B}_{t+1})) \geq 0$ if $\tilde{B}_t \leq B^*$.

4. Online Appendix I.A provides an analysis of the situation in which $\tilde{B}_t$ is not required to be interior. Farhi (2010) obtained a generalized version of this equation in an economy with capital.

From the first part of Lemma 2, $\tilde{B}(\cdot, B)$ is increasing in B and consequently $B_0 > B^*$ implies $\tilde{B}_t \geq B^*$ and $V'(\tilde{B}_t) \leq V'(B^*)$ for all $t \geq 0$. From these and equation (15), we conclude that[5]

$$V'(\tilde{B}_t) \leq \mathbb{E}V'(\tilde{B}_{t+1}). \tag{17}$$

Therefore, for $B_0 > B^*$, $V'(\tilde{B}_t)$ is a submartingale bounded above by $V'(B^*)$ and the martingale convergence theorem implies that $V'(\tilde{B}_t)$ converges almost surely. By strict concavity of V, $\tilde{B}_t$ can converge only to a level of debt B for which $E(s, B)$ is independent of s, which is possible only if $\tilde{B}_t \to B^*$ a.s. Since $\mathbb{E}R_t = \beta^{-1}$, equation (13) establishes that $\tilde{Z}_t \to Z^*$ a.s.[6] □

An insight of Proposition 2 is that the conditional covariance in equation (15) induces a drift in the stochastic process $\tilde{B}_t$ toward the risk-minimizing level of debt B^*. Here is some intuition. Fluctuations in tax rates, and therefore tax revenues, have welfare costs for reasons explained by Barro (1979). For this reason, on the margin each period the planner wants to move closer to the risk-minimizing level of debt that reduces his need to change the tax rate in response to shocks to government purchases. When p and g are perfectly correlated, fluctuations in returns on government debt $R(s)B^*$ perfectly offset fluctuations in government expenditures $g(s)$, thereby providing a perfect hedge. In this situation, the tax rate τ_t is constant in the long run.

II.B. General Case: Approximations

When p and g are imperfectly correlated, perfect hedging is impossible. To study this situation, we develop a class of second order approximations that do a good job of approximating the joint invariant distribution of government debt and tax revenues. Under particular conditions, our approximating policies are linear in shocks, a property that facilitates asymptotic analysis.

We start with the observation that random variables g and p can be expressed as

$$g(s) = \bar{g} + \sigma\epsilon_g(s), \quad p(s) = \overline{p} + \sigma\epsilon_p(s),$$

5. Although (15) assumes that $\tilde{B}_t$ is in the interior, inequality (17) is still valid when bounds on $\tilde{B}_t$ are binding.

6. When $B_0 < B^*$, $V'(\tilde{B}_t)$ is a supermartingale bounded below by $V'(B^*)$ and the same type of argument applies.

where ϵ_g and ϵ_p have mean zero but can be arbitrarily correlated with each other. We will study the properties of a Ramsey plan when shocks are small, that is, as σ approaches to 0. Let $\tilde{B}(s, B_{\_}; \sigma)$ and $\tilde{Z}(s, B_{\_}; \sigma)$ be policy functions for a given σ. Optimality conditions for problem (12) should hold for all realizations of $p(s)$, $g(s)$ and for all values of σ. Therefore first-, second-, and higher-order derivatives of those optimality conditions with respect to ϵ_g, ϵ_p, σ, assuming they exist, must all be equal to 0.[7] That insight allows us to calculate the Taylor expansion of policy rules around a current level of debt since $\tilde{B}(s, B_{\_}; 0) = B_{\_}$. In Online Appendix I.A, we show that

$$\tilde{B}(s, B_{\_}) = B_{\_} + \beta\left[g(s) - \bar{g}\right] + \left[\beta R(s) - 1\right] B_{\_} - \beta^2 \mathrm{var}(R) B_{\_} - \beta^2 \mathrm{cov}(R, g) + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2). \tag{18}$$

Here $\mathcal{O}(\sigma^3, (1-\beta)\sigma^2)$ denote all terms that appear as $\mathcal{O}(\sigma^3)$ or $(1-\beta)\mathcal{O}(\sigma^2)$.[8] The second-order expansion is linear in g and R up to terms that appear in $\mathcal{O}(\cdot)$. Since standard macroeconomic calibrations set the discount factor close to 1, we drop the $\mathcal{O}(\cdot)$ terms and proceed to study an optimal debt and tax policy implied by that approximation.[9] The linearity of the policy rules allows us to obtain a simple and transparent characterization. We show later in this section and in Section IV that this procedure provides good approximations to other more accurate approximations computed using global numerical algorithms and has the virtue of shedding light on economic principles underlying optimal debt and tax policies.

We focus on three moments: means, variances, and speeds of mean reversion to the invariant distribution of debt and taxes. We obtain these by regrouping terms in equation (18) and integrating with respect to the ergodic measure. For example, by taking unconditional expectations on both sides of equation (18), we

7. This approach is originally developed by Fleming (1971) and was applied in economics by Schmitt-Grohé and Uribe (2004). Like them, we assume that policies and value functions are sufficiently smooth that all relevant derivatives exist.

8. The term $\mathcal{O}(\sigma^3, (1-\beta)\sigma^2)$ is equivalently $\mathcal{O}(\sigma^3)$ when $\beta = 1 - \sigma$ is subject to a smoothness condition on the policy rules. Formally, setting $\beta = \bar{\beta} - \sigma$, we need that the third derivative of $\tilde{B}(s, B_{\_})$ with respect to σ is finite as $\bar{\beta} \to 1$.

9. This approximation should work well as long as average interest rates are of similar or smaller order of magnitude than the standard deviation of shocks that affect the government's budget constraint. This condition holds in our calibration to the post-World War II U.S. data in Section IV.

deduce that the unconditional mean and variance of debt can be estimated up to $\mathcal{O}(\sigma, (1-\beta))$ terms.[10]

PROPOSITION 3. The invariant distribution of $\{\tilde{B}_t, \tilde{Z}_t\}_t$ satisfies

(i) Means

$$\mathbb{E}\left(\tilde{B}_t\right) = B^* + \mathcal{O}(\sigma, 1-\beta), \quad \mathbb{E}\left(\tilde{Z}_t\right) = Z^* + \mathcal{O}(\sigma, 1-\beta);$$

(ii) Speeds of reversion to means

$$\frac{\mathbb{E}_t\left(\tilde{B}_{t+1} - B^*\right)}{\tilde{B}_t - B^*} = \frac{\mathbb{E}_0\left(\tilde{Z}_{t+1} - Z^*\right)}{\mathbb{E}_0\left(\tilde{Z}_t - Z^*\right)}$$
$$= \frac{1}{1+\beta^2 \text{var}\,(R)} + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2);$$

(iii) Variances

$$\text{var}\left(\tilde{B}_t\right) = \frac{\text{var}\,(RB^* + g)}{\text{var}\,(R)} + \mathcal{O}(\sigma, 1-\beta),$$
$$\text{var}\left(\tilde{Z}_t\right) = \left(\frac{1-\beta}{\beta}\right)^2 \text{var}\left(\tilde{B}_t\right) + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).$$

The first part of Proposition 3 shows that the risk-minimizing debt B^* is the mean of the invariant distribution, and the mean level of tax revenues is Z^*. To understand the finding that the mean of the invariant distribution of $\tilde{B}_t$ is B^*, it is useful to connect the martingale (15) to the static variance minimization problem (16). By strict concavity of the value function V, there is a one-to-one relationship between debt B_t and its marginal value to the planner, $V'(B_t)$. Inspection of the martingale equation (15) shows that the covariance term $\text{cov}_t(V'(B_{t+1}), R_{t+1})$ is important in determining the drift of the dynamics of debt in the long run. For a given B_t, the debt next period B_{t+1} depends only on E_{t+1} and consequently

$$\text{(19)} \quad \text{cov}_t(V'(B_{t+1}), R_{t+1}) \propto \text{cov}_t(E_{t+1}, R_{t+1}) + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).$$

10. Observe that while $\text{var}(R)$, $\text{var}(g)$, $\text{cov}(R, g)$ are all of order $\mathcal{O}(\sigma^2)$, functions $\frac{\text{cov}(R,g)}{\text{var}(R)}$ and $\frac{\text{var}(RB^*+g)}{\text{var}(R)}$ are of order $\mathcal{O}(1)$.

It is possible to verify that $\mathrm{cov}_t(E_{t+1}, R_{t+1}) = \frac{1}{2}\frac{\partial \mathrm{var}(R_{t+1}B_t+g_{t+1})}{\partial B_t}$. Thus, ignoring $\mathcal{O}(\sigma^3, (1-\beta)\sigma^2)$ terms, the covariance term in the martingale equation (15) is proportional to the slope of the variance of E_t with respect to government debt B_t. Since B^* minimizes variation in $E(s, B_{-})$, the slope is 0 at B^*. The change in signs of the slope implies that, to second order, $V'(B_t)$ is a submartingale when $B_t > B^*$ and supermartingale when $B_t < B^*$. Then arguments used in the proof of Proposition 2 explain why $\tilde{B}_t$ drifts toward B^*.

Proposition 3 also shows that the speed of mean reversion is determined by the variance of returns: a lower variance of returns decreases the speed of the reversion. When $B_t \neq B^*$ the fluctuations in the rate of return put additional risk into $E(s, B_t)$ that is increasing in the volatility of R and the magnitude of B_t. A more volatile R implies that it is optimal to increase the speeds at which the government should repay debt when $B_t > B^*$ and should accumulate debt when $B_t < B^*$. Dynamics of debt and taxes both approximate random walks when the security is nearly a risk-free bond, confirming an insight of Barro (1979).

The last part of Proposition 3 characterizes second moments of the invariant distribution of debt and tax rates. It shows several insights. First, the dispersion of the invariant distribution of government debt is increasing in unhedgable risk as measured by $\frac{\mathrm{var}(RB^*+g)}{\mathrm{var}(R)}$. Note that this term does not depend on σ (i.e., it is $O(1)$) and is 0 only when g and p are perfectly correlated. As long as g and p are imperfectly correlated, the variance of the invariant distribution of debt does not vanish even when σ becomes small. This outcome reflects two offsetting forces: smaller shocks imply that debt reacts less to the arrival of a shock but also that it takes longer for debt to revert to its mean. The variance of tax revenues $\tilde{Z}_t$ is determined by two considerations. Tax revenues must respond enough to changes in the level of government debt to satisfy the budget constraint, which is captured by the term $\left(\frac{1-\beta}{\beta}\right)^2 \mathrm{var}(\tilde{B}_t)$. Tax revenues also change in response directly to an expenditure shock. Since the planner wants to smooth tax rates over time, only a fraction $1-\beta$ of an innovation to government expenditures is financed by contemporaneous changes in tax revenues. Therefore, the variance induced by a contemporaneous response to aggregate shocks is of order $\mathcal{O}(\sigma^3, (1-\beta)\sigma^2)$.

Figure I illustrates the accuracy of the quadratic approximation. As a baseline we set $\beta = 0.98$ and $\gamma = 2$ and choose the joint

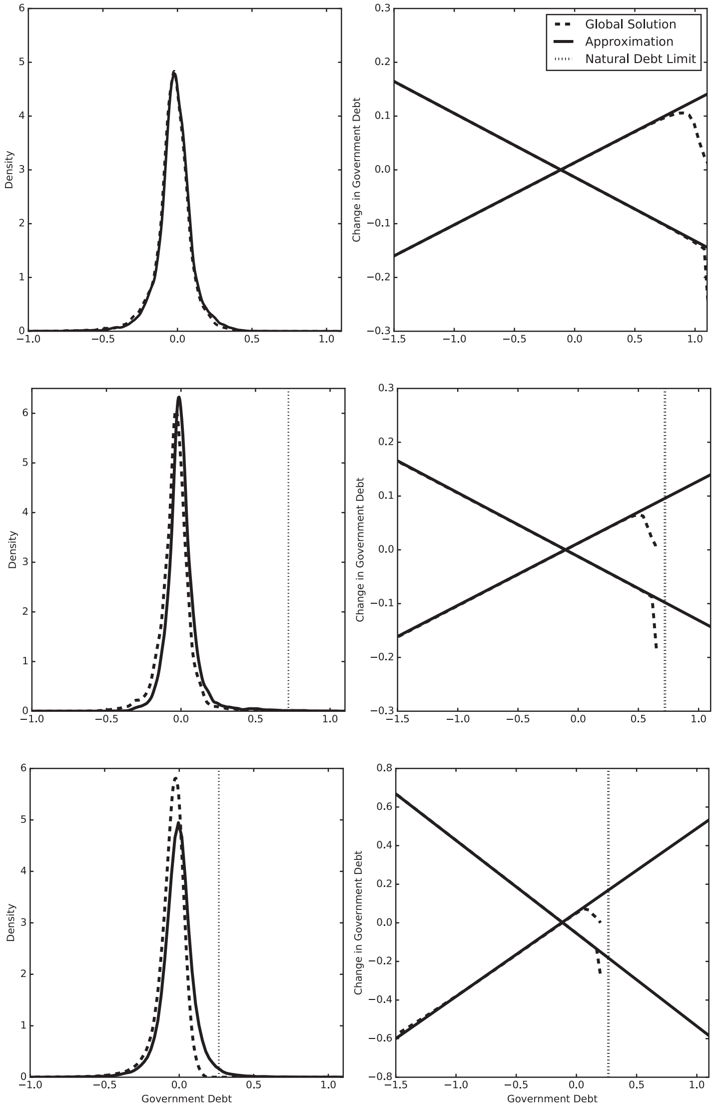

FIGURE I

Policy Rules for Debt and Ergodic Distributions

Using the quadratic approximation (solid line) and a numerical solution (dashed line), the top, middle, and bottom panels plot smoothed kernel densities (left side) and decision rules (right side) associated with baseline parameters in Table I, high discount factor ($\beta = 0.90$), and large shocks ($\sigma = 4$) settings. The right panel displays policies $\tilde{B}(s, B_{\_}) - B_{\_}$ for two values of s that correspond to the smallest and the largest pairs of $(g(s), p(s))$.

TABLE I

PARAMETERS AND MOMENTS USED FOR COMPARING THE ACCURACY OF THE QUADRATIC APPROXIMATIONS IN THE QUASILINEAR ECONOMY

Parameter	Value	Moments	Values
$\bar{g}$	0.26	Mean government expenditure relative to output	26%
std. dev. ϵ_g	0.01	Std. dev. of log government expenditures	2.6%
std. dev. $\epsilon_{\hat{p}}$	0.05	Std. dev. of returns of debt portfolio	5.1%
χ	0.67	Correlation of returns and log government expenditures	0.08

stochastic processes for (g, p) to match the standard deviation of government expenditures, returns on the government's debt portfolio and the correlation between these returns and government net-of-interest expenditures. The upper bound $\overline{B}$ is chosen to be equal to the natural debt limit that we can compute explicitly for the quasilinear setup and the lower bound is set so that the debt-to-output ratio is approximately -300%. For all the exercises we report below, we verify that $B^* \in [\underline{B}, \overline{B}]$.[11] The g, p processes are modeled as

$$g(s) = \bar{g} + \sigma\epsilon_g(s),$$
$$p(s) = 1 + \chi\sigma\epsilon_g(s) + \sigma\epsilon_{\hat{p}}(s),$$

where $\sigma = 1$ and the shocks $\epsilon_{\hat{p}}, \epsilon_g$ are finite state approximations to mean 0 normal random variables.[12] The moments we target in addition to the parameter values that achieve those targets are reported in Table I.

Given these primitives, we compute the optimal policies from first-order conditions for problem (12), and iterating on the planner's Euler equation using cubic splines as basis functions for approximating policies.[13] We then compare the outcomes of our global solution to the quadratic approximations. We plot the

11. In Section IV we do a comprehensive calibration where we match several moments of returns, output, and debt to U.S. postwar data for a richer model that allows for persistence, risk aversion, and productivity shocks. Here we use a subset of those moments to get a reasonable baseline which we modify in several directions to test the accuracy of our approximations. The details of the sample and data series used to construct these moments are in Section IV.

12. The finite state approximation ensures that $g(s) > 0$ and $p(s) > 0$ for all s.

13. Since the problem is concave, such a fixed point corresponds to the optimal policies.

invariant distribution of debt and policy rules obtained from the global solution method (dashed lines) and the quadratic approximations (solid lines) in Figure I. For parsimony, we plot policies $\tilde{B}(s, B_{-}) - B_{-}$ for two values of s that correspond to the smallest and the largest pairs of $(g(s), p(s))$. The top panels of Figure I reveal that the ergodic distribution of debt and policy functions obtained using our quadratic approximations closely resemble those obtained using the global numerical method. Our approximations differ from the global solution only at debt levels close to the natural debt limit; however, in our parameterization the ergodic distribution puts almost no mass on that region.

Proposition 3 states that our approximation errors increase with $1 - \beta$ and σ. To check how quickly these approximation errors become significant, we reduce β to 0.90 and increase σ to 4 in the middle and bottom rows of Figure I, respectively. For most of the state space, we find that the quadratic approximation continues to do well. As a consequence of the fact that our quadratic approximations assume interiority, the policies reported in the right panel display approximation errors only when debt approaches the debt limits. When $1 - \beta$ or σ is high, the quadratic approximations imply slightly higher debt than does the solution computed with numerical policy function iteration. Almost all of these differences emerge because the quadratic expansion puts positive probability on the region where debt is higher than $\bar{B}$.

III. Extensions

Forces isolated within the Section II economy prevail under alternative assumptions about motives for taxation, persistence in g and p and also fluctuations in productivity, rich sets of securities, and preferences that express aversion to consumption risk. We discuss these now.

III.A. Transfers and Redistribution

Optimal debt management in our Section II model differs significantly from that in other incomplete markets models studied by Aiyagari et al. (2002) and Farhi (2010). A key difference is that we prohibit lump-sum taxes or transfers, while Aiyagari et al. (2002) and Farhi (2010) allow positive but not negative lump-sum transfers. In our model, the invariant joint distribution of debt and taxes is unique. In the long run, debt and tax rates minimize fluctuations in gross government expenditures, including debt service requirements, $E(s, B)$. By way of contrast, optimal plans in Aiyagari et al. (2002) and Farhi (2010) have a continuum of invariant distributions of debt levels. In all of them, tax rates are 0 and debt levels are negative and big enough in absolute value to finance all net-of-interest government expenditures from earnings on the government's portfolio, and fluctuations in transfers fully absorb shocks to net-of-interest government expenditures. Here we extend our analysis to an economy with lump-sum transfers by explicitly modeling the utility enjoyed by recipients of these transfers. We show that our Section II results carry over essentially unchanged as long as the utility function of a subset of recipients of transfers who are entirely dependent on transfers is strictly concave. In that case, uncertainty about transfers is costly, prompting the government to use government debt to minimize fluctuations in both tax rates and transfers. We then discuss what drives the long-run tax rate to 0 in Aiyagari et al. (2002) and explain how to reconcile their results with ours.

A standard justification for ruling out lump-sum taxes in representative agent models is implicitly to appeal to the presence of unmodeled "poor" agents who cannot afford to pay a lump-sum tax. In this section, we study optimal anonymous transfers in an economy with such poor agents. We extend the Section II economy to have just enough heterogeneity across agents to make the analysis meaningful. In particular, we assume that in addition to a measure 1 of agents of type 1 with quasilinear preferences $U(c, l) = c - \frac{l^{1+\gamma}}{1+\gamma}$, there is a measure $n > 0$ of type 2 agents who cannot work or trade securities and who enjoy utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(c_{2,t}\right),$$

where $c_{2,t}$ is consumption of a type 2 agent in period t; U is strictly concave and differentiable on $\mathbb{R}_+$ and satisfies the Inada condition $\lim_{c\to 0} U'(c) = \infty$.

The government and type 1 agent trade the Section II security. The government imposes a linear tax rate τ_t on labor income and awards lump-sum transfers T_t that do not depend on the type of agent. Negative transfers are not feasible because a type 2 agent has no income other than transfers. Each agent receives a per capita transfer $\frac{T_t}{1+n}$. Since agent 2 lives hand to mouth, his budget constraint is

$$c_{2,t} = \frac{T_t}{1+n}.$$

The planner ranks allocations according to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\left(c_t - \frac{1}{1+\gamma} l_t^{1+\gamma} \right) + \omega U \left(c_{2,t} \right) \right],$$

for some $\omega > 0$.

The time t government budget constraint is now

$$g_t + T_t + R_t B_{t-1} = \tau_t l_t + B_t.$$

With only minimal modifications, the budget constraint of a type 1 agent, Definition 1 of a competitive equilibrium, and the Section II recursive formulation of the optimal policy problem all extend to this environment. The planner's optimal value function satisfies the Bellman equation

$$(20) \quad V(B_{-}) = \max_{Z(\cdot),B(\cdot),T(\cdot)} \int \left[\left(R(s) B_{-} - B(s) + \frac{T(s)}{1+n} \right) + \gamma \Psi(Z(s)) + \omega U\left(\frac{T(s)}{1+n} \right) + \beta V(B(s)) \right] \pi(ds),$$

subject to $Z(s) \leq \overline{Z}$, $B(s) \in [\underline{B}, \overline{B}]$, and

$$(21) \qquad Z(s) - T(s) + B(s) = R(s) B_{-} + g(s) \text{ for all } s.$$

Denoting by $\{\tilde{B}_t, \tilde{Z}_t, \tilde{T}_t\}_t$ the outcomes associated with policies that attain $V(B_{-})$ and following the same steps as in the proofs of Section II, we obtain

PROPOSITION 4. The invariant distribution of $\{\tilde{B}_t, \tilde{Z}_t, \tilde{T}_t\}_t$ is unique. The invariant distribution of $\tilde{B}_t$ satisfies properties stated in Proposition 3. The invariant distribution of $\tilde{Z}_t - \tilde{T}_t$ has the same properties as the invariant distribution of $\tilde{Z}_t$ in Proposition 3. Let $F(\tilde{T}; \omega)$ be the cumulative distribution function of the ergodic distribution of $\tilde{T}_t$. If $\omega > \omega'$ then $F(\tilde{T}; \omega)$ first-order stochastically dominates $F(\tilde{T}; \omega')$.

Insights from Section II about optimal debt management carry over to this heterogeneous economy. Fluctuations in the tax rate and (non–agent specific) lump-sum transfers now are both costly, so an optimal policy smooths both. Adjusting the tax rate in response to government expenditure shocks is costly because the deadweight loss of taxation is convex in tax rates, as stressed by Barro (1979). Adjusting transfers is also costly because that induces fluctuations in inequality.

In Aiyagari et al. (2002) and Farhi (2010) the government eventually sets tax rates to 0 and thereafter adjusts transfers one to one with government expenditures. They do not model heterogeneity explicitly but appeal to it only implicitly when they impose $T_t \geq 0$. That restriction puts a kink in the cost of using transfers: the marginal cost of an increase in transfers is 0, whereas the marginal cost of a decrease in transfers is infinite at $T_t = 0$. A high marginal cost of negative transfers creates an incentive for the governments in Aiyagari et al. (2002) and Farhi (2010) to accumulate enough assets to make the constraint $T_t \geq 0$ eventually become slack. Since fluctuations in positive transfers are costless, in the long run the government uses those transfers to offset all fluctuations in expenditures g_t.

By way of contrast, in our economy, the welfare cost of using transfers is endogenous and smooth, so that marginal costs from increasing and decreasing transfers around an optimal level $\tilde{T}_t$ are the same, $\frac{\omega}{1+n}U'(\tilde{T}_t)$; welfare costs of departing from the optimal inequality level are strictly convex. This difference accounts for the very different long-run dynamics than those discovered by Aiyagari et al. (2002).[14]

The restriction that transfers T_t are common across all types of agents is not essential for Proposition 4. Consider a slightly modified taxation scheme where the government uses a linear tax rate, meaning one with a zero intercept, for the productive type of agent and a lump-sum transfer for the unproductive types. The budget constraint of type 2 is $c_{2,t} = \frac{T_t}{n}$ and the Bellman

14. Bhandari et al. (2017b) show that this insight carries over to richer economies with much more heterogeneity and in which no agent is excluded from the financial markets.

equation (20) is altered to

$$V(B_{-}) = \max_{Z(\cdot),B(\cdot),T(\cdot)} \int \Bigg[(R(s)\,B_{-} - B(s)) + \gamma\Psi(Z(s)) + \omega U\left(\frac{T(s)}{n}\right) + \beta V(B(s))\Bigg] \pi(ds).$$

We show in Online Appendix I.B that Proposition 4 continues to hold. While the assumption that only unproductive agents receive transfers changes the average level of optimal tax revenues and the tax rate, it leaves unaffected the moments of the Ramsey policy characterized in Proposition 4.

III.B. More General Asset Structure

In this section, we study optimal management of a government's portfolio of securities by modifying the baseline Section II setup to allow $K \geq 1$ securities. Let $p^k(s)$ be the payoff of security k in state s. Each security is available in fixed net supply Q^k, which can be nonzero. Our setup thus allows for trade in financial assets like government debt and claims to Lucas trees. When Lucas trees are available, the feasibility constraint reads

$$c_t + g_t = l_t + \sum_{k=1}^{K} p_t^k Q^k.$$

To simplify, we assume that available securities consist of one period lived securities and infinitely lived consols, and that s is an i.i.d. process.[15] Let B_t^k be the government's holdings of security k at the end of period t, q_t^k its market price, and ι^k an indicator variable that is equal to 1 if security k is a consol. The government's time t budget constraint is

$$g_t + \sum_{k=1}^{K} \left(p_t^k + \iota^k q_t^k\right) \mathrm{B}_{t-1}^k = \tau_t l_t + \sum_{k=1}^{K} q_t^k \mathrm{B}_t^k.$$

Let $R_t^k = \frac{p_t^k + \iota^k q_t^k}{q_{t-1}^k}$ be the gross return on security k, and let $B_t^k = q_t^k \mathrm{B}_t^k$ be the market value of holdings of security k so that we can write

15. The extension of our results to arbitrary finite period securities is straightforward but requires additional notation. Extensions to richer shock processes follow along the lines of Section III.C.

this budget constraint as

$$g_t + \sum_{k=1}^{K} R_t^k B_{t-1}^k = \tau_t l_t + \sum_{k=1}^{K} B_t^k.$$

Let $B_t \equiv \sum_{k=1}^{K} B_t^k$ be the market value of the government's portfolio. We restrict holdings B_t to be in a compact set $[\underline{B}, \overline{B}]$. We assume that these bounds are sufficiently large so that the risk-minimizing portfolio B^* to be defined below is feasible. Without loss of generality, we assume that no security is redundant in the sense that the vectors $\{R^k\}_{k=1}^{K}$ are linearly independent. We use $\mathbf{R}(s)$ to denote returns $(R^1(s), ..., R^K(s))$, $\mathbf{B}$ and $\mathbf{1}$ to denote a K dimensional (column) vector of security holdings and of ones, respectively. Let $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ and $\mathbb{C}[\mathbf{R}, g]$ be a matrix of the covariances of returns and a vector of covariances of returns with government purchases g, respectively. When the matrix $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is nonsingular, we define

$$B^* \equiv -\mathbf{1}^{\intercal}\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1}\mathbb{C}[\mathbf{R}, g], \tag{22}$$

which, as we show below, is the risk-minimizing level of government debt that generalizes equation (16) to the case of multiple assets. Whenever B^* is well defined, we also define $Z^* \equiv \frac{1-\beta}{\beta} B^* + \bar{g}$.

Temporarily suppose that government portfolio weights are fixed, meaning that there exist constants $\psi_1, ..., \psi_K$ such that $\frac{B_t^k}{\sum_k B_t^k} = \psi_k$ for all t. Define $R(s) \equiv \sum_{k=1}^{K} \psi_k R^k(s)$. Then the optimal policy problem is equivalent to the one in Section II. Thus, if the government arbitrarily fixes its portfolio weights then, subject to that arbitrary choice, all Section II insights about optimal debt management and fiscal policy still apply.

Now suppose that the Ramsey planner optimally chooses government portfolio weights each period. The Ramsey problem in this case can be written recursively with the end of period market value of the government's portfolio the only state variable in the planner's value function:

$$\begin{aligned} V(B_{-}) = \max_{Z(\cdot), B(\cdot), \mathbf{B}} \int \Big[\big(\mathbf{R}(s)^{\intercal}\,\mathbf{B} - B(s)\big) + \gamma \Psi(Z(s)) \\ + \beta V(B(s)) \Big] \pi(ds), \end{aligned} \tag{23}$$

where maximization is subject to $Z(s) \leq \overline{Z}$, $B(s) \in [\underline{B}, \overline{B}]$, $\mathbf{1}^\intercal \mathbf{B} = B_{-}$, and

$$B(s) + Z(s) = \mathbf{R}(s)^\intercal \mathbf{B} + g(s) \quad \text{for all } s. \tag{24}$$

We first establish that:

LEMMA 3. Problem (23) has a unique solution. If $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is nonsingular or if g is not in the span of $\mathbf{R}$, then the invariant distribution generated by policies that attain $V(B_{-})$ is unique; if $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is singular and g is in the span of $\mathbf{R}$, then optimal policies satisfy $\tilde{B}(s, B_{-}) = B_{-}$ for all s and $Z(s, B_{-})$ is independent of s.

Lemma 3 extends Proposition 1 and Proposition 3 for $K > 1$ assets. An additional insight of Lemma 3 is that if $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is singular and g is in the span of $\mathbf{R}$, then the complete market Ramsey allocation can be attained by the planner.

We turn to quadratic approximations to characterize the moments of the invariant distribution of total debt and tax rates. As in the Section II baseline model, we scale the volatility of all shocks by σ and take a second-order Taylor expansion of the policies $\tilde{Z}(s, B_{-}; \sigma)$, $\tilde{B}(s, B_{-}; \sigma)$, and $\tilde{\mathbf{B}}(B_{-}; \sigma)$ around $\sigma = 0$.[16] At $\sigma = 0$ the portfolio problem is indeterminate, but the next lemma shows that there is a unique limiting portfolio as $\sigma \to 0$ that solves a variance minimization problem.

LEMMA 4. $\lim_{\sigma \to 0} \tilde{\mathbf{B}}(B_{-}; \sigma) = \mathbf{B}^*(B_{-})$ where $\mathbf{B}^*(B_{-})$ solves

$$\min_{\mathbf{B}} \operatorname{var}\left(\sum_k B^k R^k + g\right) \quad \text{subject to} \quad \mathbf{1}^\intercal \mathbf{B} = B_{-}. \tag{25}$$

We can relate B^* in equation (22) to $\mathbf{B}^*(\cdot)$ defined in Lemma 4. Suppose that the matrix $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is nonsingular. Then the constraint $\mathbf{1}^\intercal \mathbf{B} = B_{-}$ in equation (25) binds, making the risk-minimizing variance depend on the level of debt B_{-}. This variance is minimized at $B_{-} = B^*$, making B^* the risk-minimizing debt level, which satisfies $B^* = \mathbf{1}^\intercal \mathbf{B}^*(B^*)$.[17]

16. We continue to assume that policies are sufficiently differentiable.

17. If $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is singular, then the constraint $\mathbf{1}^\intercal \mathbf{B} = B_{-}$ does not bind, and any debt level is risk-minimizing.

As in Section II, we show that the B^* is the long-run mean of the second-order approximation to the optimal policy for the market value of the government debt portfolio. When $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is nonsingular, the Taylor expansion around $\mathbf{B}^*(B_-)$ yields

$$\begin{aligned}\tilde{B}(s, B_-) = {} & B_- + \beta\left[g(s) - \bar{g}\right] + \left[\beta\mathbf{R}(s) - \mathbf{1}\right]^\intercal \mathbf{B}^*(B_-)\\ & - \frac{\beta^2 B_-}{\mathbf{1}^\intercal\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1}\mathbf{1}} - \frac{\beta^2\mathbf{1}^\intercal\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1}\mathbb{C}[\mathbf{R}, g]}{\mathbf{1}^\intercal\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1}\mathbf{1}}\\ & + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).\end{aligned}$$

When $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is singular,

$$\begin{aligned}\tilde{B}(s, B_-) = {} & B_- + \beta\left[g(s) - \bar{g}\right] + \beta\left[\mathbf{R}(s) - \mathbf{1}\right]^\intercal \mathbf{B}^*(B_-)\\ & + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).\end{aligned} \tag{26}$$

PROPOSITION 5. Suppose that $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is nonsingular. The invariant distribution of $\{\tilde{B}_t, \tilde{Z}_t\}_t$ has

(i) Means

$$\mathbb{E}\left(\tilde{B}_t\right) = B^* + \mathcal{O}(\sigma, 1-\beta), \quad \mathbb{E}\left(\tilde{Z}_t\right) = Z^* + \mathcal{O}(\sigma, 1-\beta);$$

(ii) Speeds of mean reversions

$$\begin{aligned}\frac{\mathbb{E}_t\left(\tilde{B}_{t+1} - B^*\right)}{\tilde{B}_t - B^*} &= \frac{\mathbb{E}_0\left(\tilde{Z}_{t+1} - Z^*\right)}{\mathbb{E}_0\left(\tilde{Z}_t - Z^*\right)}\\ &= \frac{\beta^{-2}\mathbf{1}^\intercal\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1}\mathbf{1}}{1 + \beta^{-2}\mathbf{1}^\intercal\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1}\mathbf{1}} + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2);\end{aligned}$$

(iii) Variances

$$\begin{aligned}\operatorname{var}\left(\tilde{B}_t\right) &= \left(\mathbf{1}^\intercal\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1}\mathbf{1}\right)\operatorname{var}\left(-\mathbb{C}[\mathbf{R}, g]^\intercal\mathbb{C}[\mathbf{R}, \mathbf{R}]^{-1}\mathbf{R} + g\right)\\ &\quad + \mathcal{O}(\sigma, 1-\beta),\\ \operatorname{var}\left(\tilde{Z}_t\right) &= \left(\frac{1-\beta}{\beta}\right)^2\operatorname{var}\left(\tilde{B}_t\right) + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).\end{aligned}$$

Government holdings of individual securities satisfy

$$
\begin{aligned}
(27)\quad \tilde{\mathbf{B}}_t &= \mathbf{B}^*\left(\tilde{B}_t\right) + \mathcal{O}(\sigma^2, (1-\beta)\sigma) \\
&= -\mathbb{C}[\mathbf{R},\mathbf{R}]^{-1}\mathbb{C}[\mathbf{R},g] + \frac{\mathbb{C}[\mathbf{R},\mathbf{R}]^{-1}\mathbf{1}}{\mathbf{1}^\intercal\mathbb{C}[\mathbf{R},\mathbf{R}]^{-1}\mathbf{1}} \\
&\quad \times \left(\tilde{B}_t + \mathbf{1}^\intercal\mathbb{C}[\mathbf{R},\mathbf{R}]^{-1}\mathbb{C}[\mathbf{R},g]\right) + \mathcal{O}(\sigma^2, (1-\beta)\sigma).
\end{aligned}
$$

Some examples illustrate these findings.

EXAMPLE 1. Suppose that there are two securities with $\mathrm{var}(R^k) > 0$ for $k = 1, 2$ and that the return on security 1 is perfectly correlated with g while the return on security 2 is orthogonal to the return on security 1. Then Proposition 5 implies that the ergodic mean of the value of the government's debt portfolio is $B^* = -\frac{\mathrm{cov}(R^1,g)}{\mathrm{var}(R^1)}$, that the speed of convergence to B^* is $(1+\beta^2\frac{\mathrm{var}(R^2)}{\mathrm{var}(R^1)+\mathrm{var}(R^2)}\mathrm{var}(R^1))^{-1}$, and that its ergodic variance is 0. From equation (27), the optimal portfolio along transition paths satisfies

$$
\tilde{B}^1(\tilde{B}_t) = \frac{\mathrm{var}\left(R^2\right)}{\mathrm{var}\left(R^1\right)+\mathrm{var}\left(R^2\right)}\tilde{B}_t + \frac{\mathrm{var}\left(R^1\right)}{\mathrm{var}\left(R^1\right)+\mathrm{var}\left(R^2\right)}B^*
$$

$$
\tilde{B}^2(\tilde{B}_t) = \frac{\mathrm{var}\left(R^1\right)}{\mathrm{var}\left(R^1\right)+\mathrm{var}\left(R^2\right)}\tilde{B}_t - \frac{\mathrm{var}\left(R^1\right)}{\mathrm{var}\left(R^1\right)+\mathrm{var}\left(R^2\right)}B^*,
$$

with $\tilde{B}^2(\tilde{B}_t) \to 0$ a.s.

Complete hedging can be achieved with the government holding security 1 only, just as in Proposition 2, so that holding any security 2 is suboptimal asymptotically. If the market value of the initial government debt does not equal B^*, it is optimal to invest in security 2 along the transition path because doing this reduces risk for the government until the steady state is reached. As a result, noting that $\frac{\mathrm{var}(R^2)}{\mathrm{var}(R^1)+\mathrm{var}(R^2)} < 1$, the speed of convergence to the long-run portfolio is slower than when only security 1 can be traded.

EXAMPLE 2. Consider a setting with two securities whose payoffs are perfectly correlated with g and $0 \leq \text{var}(R^1) < \text{var}(R^2)$.[18] There exist unique constants ψ_1, ψ_2, ξ_1, and ξ_2 such that

$$\psi_1 R^1(s) + \psi_2 R^2(s) = g(s)$$

and

$$\xi_1 R^1(s) + \xi_2 R^2(s) = \frac{1}{\beta}.$$

Note that $\psi_1 + \psi_2 = \beta\bar{g}$ and $\xi_1 + \xi_2 = 1$. Now the covariance matrix $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is singular. The risk-minimizing portfolio satisfies $B^{*,k}(B_{\_}) = (B_{\_} + \beta\bar{g})\xi_k - \psi_k$. Holding it allows the government to attain complete markets allocations for any $B_{\_}$; the value of government debt equal its initial value for all $t \geq 0$.

It is instructive to study how an optimal portfolio changes as R^2 approaches R^1. For simplicity, suppose that $R^1(s) = \frac{1}{\beta}$ and $R^2(s) = \frac{1}{\beta} - \varepsilon(g(s) - \bar{g})$. For a given $B_{\_}$, optimal asset positions are $B^{*,2} = \frac{1}{\varepsilon}$ and $B^{*,1} = B_{\_} - B^{*,2}$, both of which become arbitrarily large as $\varepsilon \to 0$. This outcome explains why Buera and Nicolini (2004) and Farhi (2010) found that the government should take extremely large asset positions to hedge its risk. Those papers allowed a planner to trade a risk-free one-period security plus other securities (long bonds in Buera and Nicolini 2004, capital in Farhi 2010). The returns on those securities had low volatilities and high correlations with government expenditures. Consistent with our example, those authors found that an optimal portfolio has huge positions in these securities.

EXAMPLE 3. Suppose that $\text{cov}(R^k, R^l) = 0$ for all $k \neq l$. Now $\mathbb{C}[\mathbf{R}, \mathbf{R}]$ is a diagonal matrix and $\frac{\partial \tilde{B}^k(B_{\_})}{\partial B_{\_}} \propto \frac{1}{\text{var}(R^k)}$ from equation (27). As the value of the outstanding government debt increases, its optimal composition shifts toward securities that have lower variances of returns. In the limit, as the variance of returns of one security approaches 0, all of the adjustments to changes in $B_{\_}$ use that security.

18. Note that risk-free returns are in the closure of the set of returns that are perfectly correlated with g. We follow a convention of calling a risk-free security to be perfectly correlated with g.

We can use this example to construct a simple model of an optimal maturity structure of government debt. Suppose that the government can issue a one-period risk-free bond and a consol with a stochastic coupon. Then the optimal issue of the consol is $\tilde{B}_t^2 = -\frac{\mathrm{cov}(R^2,g)}{\mathrm{var}(R^2)}$, which is independent of $\tilde{B}_t$, while the optimal issue of the riskless security is $\tilde{B}_t^1 = \tilde{B}_t - \tilde{B}_t^2$. Hence, the optimal effective maturity $\frac{\tilde{B}_t^2}{\tilde{B}_t^1}$ of government debt is decreasing in the value of outstanding debt $\tilde{B}_t$.

III.C. More General Shock Processes

In this section, we modify the Section II baseline model setup to include richer shock processes. In addition to expenditure and payoff shocks, we introduce fluctuations in productivity θ and allow g, p, θ to be correlated across time and with each other.[19]

We follow the set up of Section II but assume that state $s = (p, g, \theta)$ follows a first-order Markov process. The conditional probability density of s_t is described by a Markov kernel $\pi(\cdot|s_{-})$, where s_{-} is the realization of the shock in period $t-1$. We assume that π has a unique invariant measure λ. The feasibility constraint now takes the form

$$c_t + g_t = \theta_t l_t, \tag{28}$$

and the return in state s is $R(s, s_{-}) = \frac{p(s)}{\beta \int p(s')\pi(ds'|s_{-})}$.

Let $\Theta \equiv \theta^{\frac{1+\gamma}{\gamma}}$ and $Z \equiv \tau(1-\tau)^{\frac{1}{\gamma}}$. As in Section II, there is a one-to-one correspondence between Z and τ for $Z \leq \bar{Z}$. The tax revenues with productivity shocks are equal to ΘZ. Let $\bar{g}$ and $\bar{\Theta}$ denote ergodic means of g and Θ. Let $\Omega(Z, s) \equiv \frac{l^{1+\gamma}(Z,s)}{1+\gamma}$, where the function $l^{1+\gamma}(Z, s)$ is now defined by

$$\Theta(s) Z = \Theta(s)^{\frac{\gamma}{1+\gamma}} l(Z, s) - l^{1+\gamma}(Z, s).$$

19. In Online Appendix II we show that discount factor shocks as in Albuquerque et al. (2016) are isomorphic to an economy with redefined g, p, θ shocks and no discount factor shocks.

Following Section II arguments, the Ramsey planner's value function satisfies the Bellman equation

$$(29)\quad V\left(B_{-}, s_{-}\right)=\max_{Z(\cdot), B(\cdot)} \int\left[\left(R(s, s_{-}) B_{-}-B(s)\right)+\gamma \Omega\left(Z(s), s\right)\right.$$
$$\left.+\beta V\left(B(s), s\right)\right] \pi\left(d s \mid s_{-}\right),$$

where maximization is subject to $Z(s) \leq \overline{Z},\ B(s) \in [\underline{B}, \overline{B}]$ and

$$(30)\qquad B(s)=R(s, s_{-}) B_{-}+g(s)-\Theta(s) Z(s) \text{ for all } s.$$

In the interior, optimal debt satisfies

$$(31)\quad V\left(\tilde{B}_t, s_t\right)=\mathbb{E}_t V\left(\tilde{B}_{t+1}, s_{t+1}\right)+\beta \operatorname{cov}_t(R_{t+1}, V'(\tilde{B}_{t+1}, s_{t+1})),$$

which extends the martingale equation (15) to persistent shocks.

As a counterpart to expression (16), we now define the risk-minimizing government debt B^* for the general case being studied here. As before, the Ramsey planner chooses government debt to minimize risk and fluctuations in the tax rate. The shocks here introduce additional considerations not present in the Section II baseline model. First, fluctuations in productivity imply that tax revenues are stochastic even when the tax rate is constant. Fix the tax rate at level τ and observe that primary deficit X_τ, defined as the difference between expenditures and tax revenues, is

$$(32)\qquad X_\tau(s)=g(s)-\Theta(s)(1-\tau)^{\frac{1}{\gamma}} \tau.$$

Fluctuations in the primary government deficit are driven by shocks to both government expenditures and productivity. Furthermore, when these processes are persistent, the current state s_t conveys information about future primary deficits. Now government debt will play an important role in hedging fluctuations in the expected present value of primary deficits.

For a random variable $x(s)$ that is a function of the current state only, a discounted present value of x conditional on s is $PV(x; s) \equiv \mathbb{E}[\sum_{t=0}^{\infty} \beta^t x_t | s_0 = s]$. Since the planner keeps the tax rate approximately constant, the mean of the invariant distribution for debt and the level of tax rate are linked through the government budget constraint by

$$(33)\qquad \left(\frac{1-\beta}{\beta}\right) B=\bar{g}-\bar{\Theta}(1-\tau)^{\frac{1}{\gamma}} \tau,$$

which defines an implicit function $\tau(B)$. We define B^* as the level of debt that best hedges fluctuations in $PV(X_{\tau(B)}; s)$:

$$B^* \equiv \arg\min_B \operatorname{var}\left[RB + PV(X_{\tau(B)})\right]. \tag{34}$$

We define Z^* as $Z^* \equiv \frac{1}{\bar{\Theta}}[\bar{g} + \frac{1-\beta}{\beta}B^*]$.

We again use a second-order approximation of policies to show that B^* is the long-run target level of government debt. To state things compactly, it helps to define two mappings. For a pair of random variables $x(s, s_-)$, $y(s, s_-)$, the covariance conditional on s_- is

$$\mathcal{C}^{x,y}(s_-) \equiv \int x(s, s_-)y(s, s_-)\pi(ds|s_-) - \left(\int x(s, s_-)\pi(ds|s_-)\right) \times \left(\int y(s, s_-)\pi(ds|s_-)\right)$$

and the conditional mean of $x(s)$ is

$$\mathrm{E}(x; s_-) \equiv \int x(s, s_-)\pi(ds|s_-).$$

Note that both $\mathcal{C}^{x,y}(\cdot)$ and $\mathrm{E}(x;\cdot)$ are random variables on S. Taking a Taylor expansion of optimal policies that attain the optimal value function $V(B_-, s_-)$ that satisfies Bellman equation (29) along lines taken in Section II we get[20]

$$\begin{aligned}\tilde{B}(s, B_-, s_-) = B_- &+ \left[g(s) - (1-\beta)PV(g; s)\right] \\ &- \bar{\Phi}(B_-)\left[\Theta(s) - (1-\beta)PV(\Theta; s)\right] + B_-\left[\beta R(s, s_-) - 1\right] \\ &- (1-\beta)\beta^2\left[B_- PV\left(\mathcal{C}^{R,R}; s\right) + PV\left(\mathcal{C}^{R,PV(g)}; s\right)\right. \\ &\left.- \bar{\Phi}(B_-)PV\left(\mathcal{C}^{R,PV(\Theta)}; s\right)\right] + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2),\end{aligned} \tag{35}$$

where $\bar{\Phi}(B_-) = \frac{1}{\bar{\Theta}}[(\frac{1-\beta}{\beta})B_- + \bar{g}]$.

The first line on the right side of equation (35) collects first-order expansion terms that capture direct effects of shocks to g, p, θ on the asset positions. Government debt increases if the current

20. A detailed derivation of equation (35) appears in Online Appendix I.D.

realization of g is greater than annuitized expected future expenditures, $(1 - \beta)PV(g;\, s)$; if current realization of productivity is less than annuitized expected future productivity $(1 - \beta)PV(\Theta; s)$; or if the interest payments on debt are unexpectedly high. These terms express how optimal policy uses debt to smooth aggregate shocks and embody principles conveyed by Barro (1979). The second line on the right side of equation (35) collects second-order terms that consist of conditional variances and covariances of the return with expenditure and productivity shocks, which capture the hedging motives of the government.

It is convenient to rewrite equation (35) in terms of ergodic moments of $(g,\, p,\, \theta)$. For a random variable $x(s, s_{-})$, let $\mathbb{E}x = \int \int x(s, s_{-})\pi(ds|s_{-})\lambda(ds_{-})$ be its ergodic mean. Similarly, let var (x) and cov (x, y) denote ergodic variances and covariances of random variables x and y, respectively. In Online Appendix I.D, we show that under our assumption about π, we can write equation (35) as

(36)
$$\begin{aligned}\tilde{B}(s, B_{-}, s_{-}) = B_{-} &+ \left[g(s) - (1-\beta)\, PV(g; s)\right]\\ &- \bar{\Phi}(B_{-})\left[\Theta(s) - (1-\beta)\, PV(\Theta; s)\right] + B_{-}\left[\beta R(s, s_{-}) - 1\right]\\ &- \beta^2 \left[B_{-}\mathrm{var}(R) + \mathrm{cov}(R, PV(g))\right.\\ &\left. - \bar{\Phi}(B_{-})\mathrm{cov}(R, PV(\Theta))\right] + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).\end{aligned}$$

For a random variable $x(s, s_{-})$, let $\hat{x}(s, s_{-}) \equiv x(s, s_{-}) - \mathrm{E}(x; s_{-})$. We can use equation (36) to obtain

PROPOSITION 6. The invariant distribution of $\{\tilde{B}_t, \tilde{Z}_t\}_t$ has

(i) Means

$$\mathbb{E}\left(\tilde{B}_t\right) = B^{*} + \mathcal{O}(\sigma, 1-\beta), \quad \mathbb{E}\left(\tilde{Z}_t\right) = Z^{*} + \mathcal{O}(\sigma, 1-\beta);$$

(ii) Speeds of reversion to means

$$\begin{aligned}\frac{\mathbb{E}_t\left(\tilde{B}_{t+1} - B^{*}\right)}{\tilde{B}_t - B^{*}} &= \frac{\mathbb{E}_0\left(\tilde{Z}_{t+1} - Z^{*}\right)}{\mathbb{E}_0\left(\tilde{Z}_t - Z^{*}\right)}\\ &= \frac{1}{1+\beta^2 \mathrm{var}\,(R)} + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2);\end{aligned}$$

(iii) Variances Define $\mathbb{B}(s) \equiv B^* - \beta[\mathrm{E}(PV(g-\bar{g};s');s) - \bar{\Phi}(B^*)\mathrm{E}(PV(\Theta-\bar{\Theta};s');s)]$. Then

$$\operatorname{var}\left(\tilde{B}_t - \mathbb{B}_t\right) = \frac{\operatorname{var}\left(\hat{PV}(g) - \bar{\Phi}(B^*)\hat{PV}(\Theta) + \mathbb{B}\hat{R}\right)}{\operatorname{var}(R)} + \mathcal{O}(\sigma, 1-\beta),$$

$$\operatorname{var}\left(\tilde{Z}_t\right) = \left(\frac{1-\beta}{\beta}\right)^2 \operatorname{var}\left(\tilde{B}_t - \mathbb{B}_t\right) + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).$$

Proposition 6 shows that just as in Section II, the Ramsey planner chooses government debt to minimize risk and keep the tax rate approximately constant. One can extend our approximations (19) to show that the Euler equation (31) induces reversion of government debt to a risk-minimizing level. Productivity shocks now induce fluctuations in tax revenues even when the tax rate is constant.

The risk-minimizing debt level B^* can be computed from formula (34) and further simplified after we observe that $\frac{\partial}{\partial B}\tau(B) = \mathcal{O}(1-\beta)$. Given this, we have

$$B^* = -\frac{\operatorname{cov}\left(R, PV(X_{\tau(B)})\right)}{\operatorname{var}(R)} + \mathcal{O}(\sigma, 1-\beta) \text{ for any } B$$

$$(37) \qquad = -\frac{\operatorname{cov}\left(R, PV(g)\right) - \frac{\bar{g}}{\bar{\Theta}}\operatorname{cov}\left(R, PV(\Theta)\right)}{\operatorname{var}(R)} + \mathcal{O}(\sigma, 1-\beta).$$

The simple formula (37) for the approximate risk-minimizing debt level presents a further insight that we shall exploit in Sections III.D and IV. It shows that the endogenous covariances that appear in this formula are not very sensitive to values of $\tau(B)$ at which they are evaluated. That suggests that if we were to observe data generated under a suboptimal tax rate policy $\tau(B)$, observations of the primary deficit $X_{\tau(B)}$ would still allow us to compute the optimal level of debt B^* accurately by using equation (37).

We end this section by applying our formulas when g, Θ, and p obey the AR(1) processes

$$g_t = (1-\rho_g)\bar{g} + \rho_g g_{t-1} + \varepsilon_{g,t},$$

$$\Theta_t = (1-\rho_\Theta)\bar{\Theta} + \rho_\Theta \Theta_{t-1} + \varepsilon_{\Theta,t}$$

$$p_t = \bar{p} + \varepsilon_{p,t},$$

where $\varepsilon_{g,\,t}$, $\varepsilon_{p,t}$, $\epsilon_{\Theta,t}$ are i.i.d. over time with zero means. Now $\text{cov}(R, PV(g)) = \frac{\text{cov}(R,g)}{1-\rho_g\beta}$ and $\text{cov}(R, PV(\Theta)) = \frac{\text{cov}(R,\Theta)}{1-\rho_\Theta\beta}$. Therefore

$$B^* = -\left(\frac{1}{1-\rho_g\beta}\right)\frac{\text{cov}(R,g)}{\text{var}(R)} + \frac{\bar{g}}{\bar{\Theta}}\left(\frac{1}{1-\rho_\Theta\beta}\right)\frac{\text{cov}(R,\Theta)}{\text{var}(R)},$$

$$(38) \quad = -\left(\frac{\beta}{1-\rho_g\beta}\right)\frac{\text{cov}(\epsilon_p,\epsilon_g)}{\text{var}(\epsilon_p)} + \frac{\bar{g}}{\bar{\Theta}}\left(\frac{\beta}{1-\rho_\Theta\beta}\right)\frac{\text{cov}(\epsilon_p,\epsilon_\Theta)}{\text{var}(\epsilon_p)}.$$

Equation (38) shows how autocorrelations affect the target level of government debt. For instance, keeping ρ_Θ fixed, higher persistence of the expenditure shocks as measured by ρ_g implies a higher absolute value of government debt asymptotically. The sign of the covariance between returns and the primary government deficit determines the sign of the mean level of government debt.

III.D. Risk Aversion and Endogenous Returns

We extend our analysis to a setting in which the representative agent has preferences that display risk aversion. We retain other assumptions of Section III.C but now allow curvature in the utility of consumption by assuming that preferences are described by

$$(39) \qquad U(c,l) = \frac{c^{1-\alpha}-1}{1-\alpha} - \frac{l^{1+\gamma}}{1+\gamma}.$$

We let $U_{x,t}$ or $U_{xy,\,t}$ denote first and second derivatives of U with respect to x, $y \in \{c, l\}$. We assume that natural debt limits restrict the consumer, which ensures that first-order conditions are satisfied off corners.

An allocation $\{c_t,\ l_t,\ \mathrm{B}_t\}_t$ is a competitive equilibrium if and only if it satisfies the feasibility constraint (28) and implementability conditions

(40)
$$U_{c,t}\mathrm{B}_t + U_{c,t}\left[\theta_t l_t + \frac{U_{l,t}}{U_{c,t}}l_t - g_t\right] = \frac{p_t U_{c,t}}{\beta\mathbb{E}_{t-1}p_t U_{c,t}}U_{c,t-1}\mathrm{B}_{t-1} \quad t \geq 1,$$

$$(41) \qquad c_0 + b_0 = -\frac{U_{l,0}}{U_{c,0}}l_0 + p_0\beta^{-1}\mathrm{B}_{-1}.$$

An optimal allocation maximizes $\mathbb{E}_0 \sum_t \beta^t U(c_t, l_t)$ subject to constraints (28), (40), and (41).

It is helpful to redefine variables. Let $\mathcal{B}_t \equiv U_{c,t}\mathrm{B}_t$, $\mathcal{R}_t \equiv \frac{U_{c,t} p_t}{\beta \mathbb{E}_{t-1} U_{c,t} p_t}$, and $\mathcal{X}_t \equiv U_{c,t}[g_t - \tau_t \theta_t l_t]$ be marginal utility adjusted debt, return, and primary deficit. Using the household's first-order necessary conditions and the resource constraint, at any state s for a given tax rate τ, a household's consumption $c_\tau(s)$ satisfies

$$(1-\tau)\,\theta(s) c_\tau(s)^{-\alpha} + \left(\frac{c_\tau(s) + g(s)}{\theta(s)}\right)^{\gamma} = 0. \tag{42}$$

Along any history (s^{t-1}, s_t) effective returns and effective deficits can be expressed in terms of exogenous states s_t and a period t tax rate τ as

$$\mathcal{R}_\tau\left(s_t, s^{t-1}\right) = \frac{c_{\tau(s_t)}\left(s_t\right)^{-\alpha} p\left(s_t\right)}{\beta \int c_{\tau(s')}\left(s'\right)^{-\alpha} p\left(s'\right) \pi\left(ds' | s_{t-1}\right)},$$

$$\mathcal{X}_\tau\left(s_t, s^{t-1}\right) = \left(\frac{c_\tau\left(s_t\right) + g\left(s_t\right)}{\theta\left(s_t\right)}\right)^{1+\gamma} - c_\tau\left(s_t\right)^{1-\alpha}.$$

These transformations allow us to assert that the Ramsey planner's optimal value function for $t \geq 1$ satisfies the Bellman equation:

$$V\left(\mathcal{B}_-, s_-\right) = \max_{\tau(\cdot), \mathcal{B}(\cdot)} \int \left[U\left(c_{\tau(s)}(s), \frac{c_{\tau(s)}(s) + g(s)}{\theta(s)}\right) + \beta V\left(\mathcal{B}(s), s\right)\right] \pi\left(ds | s_-\right), \tag{43}$$

where maximization is subject to

$$\mathcal{B}(s) = \mathcal{R}_\tau\left(s, s_-\right) \mathcal{B}_- + \mathcal{X}_{\tau(s)}\left(s\right) \text{ for all } s. \tag{44}$$

Problem (43) closely resembles problem (29) except that all variables have been transformed into their effective counterparts.[21]

21. The planner's problem at $t = 0$ at initial debt B_{-1} and state s_{-1} is

$$\max_{\tau(\cdot), \mathcal{B}(\cdot)} \int \left[U\left(c_{\tau(s)}(s), \frac{c_{\tau(s)}(s) + g(s)}{\theta(s)}\right) + \beta V\left(\mathcal{B}(s), s\right)\right] \pi\left(ds | s_{-1}\right).$$

subject to

$$\mathcal{B}(s_0) = \mathcal{X}_\tau\left(s\right) + U_c\left(c_{\tau(s)}(s)\right) p(s) \beta^{-1} \mathrm{B}_{-1} \quad \forall s.$$

The planner now uses effective debt to smooth risk, and the evolution of the optimal effective government debt level satisfies

$$V'\left(\tilde{\mathcal{B}}_t, s_t\right) = \mathbb{E}_t V'\left(\tilde{\mathcal{B}}_{t+1}, s_{t+1}\right) + \beta \mathrm{cov}_t(\mathcal{R}_{t+1}, V'(\tilde{\mathcal{B}}_{t+1}, s_{t+1})),$$

an analogue of equation (31). The economic intuition for this equation is that the planner still uses the covariance of returns with the shadow cost of debt to hedge risk, but adjusts all the variables for the shadow costs of raising revenues.

We can use insights from Section III.C to define a risk-minimizing level of effective debt as

$$\mathcal{B}^* \equiv \arg\min_{\mathcal{B}} \mathrm{var}\left[\mathcal{R}\mathcal{B} + PV(\mathcal{X}_{\tau(\mathcal{B})})\right], \tag{45}$$

where $\tau(\mathcal{B})$ satisfies the following ergodic version of the government budget constraint

$$\left(\frac{1-\beta}{\beta}\right)\mathcal{B} = \mathbb{E}\mathcal{X}_\tau\left(\cdot\right). \tag{46}$$

We extend Proposition 6 to accommodate risk-averse preferences.

PROPOSITION 7. The ergodic mean and the speed of mean reversion of effective debt $\{\tilde{\mathcal{B}}_t\}_t$ are

$$\mathbb{E}\tilde{\mathcal{B}}_t = \mathcal{B}^* + \mathcal{O}(\sigma, 1-\beta),$$

$$\frac{\mathbb{E}_t\left(\tilde{\mathcal{B}}_{t+1} - \mathcal{B}^*\right)}{\tilde{\mathcal{B}}_t - \mathcal{B}^*} = \frac{1}{1+\beta^2\mathrm{var}(\mathcal{R}_{\tau(\mathcal{B}^*)})} + \mathcal{O}(\sigma^3, (1-\beta)\sigma^2).$$

Furthermore, $\mathcal{B}^*$ satisfies

$$\mathcal{B}^* = -\frac{\mathrm{cov}\left(\mathcal{R}_{\tau(\mathcal{B})}, PV\left(\mathcal{X}_{\tau(\mathcal{B})}\right)\right)}{\mathrm{var}\left(\mathcal{R}_{\tau(\mathcal{B})}\right)} + \mathcal{O}(1-\beta) \text{ for any } \mathcal{B}. \tag{47}$$

Proposition 7 confirms our theme that a target level of government debt under the optimal plan solves a variance-minimization problem. It also extends a finding from equation (37) that although second moments of returns and the primary government deficit depend on government policy, effects on the tax rate are small, so omitting them lead to errors of order only $\mathcal{O}(1-\beta)$.

Equation (47) has implications about an optimal level of risk-free debt that relate to findings of Aiyagari et al. (2002). The return on risk-free debt is known one period in advance, but the effective return is not. In particular, the effective return is high in states in which consumption is low, namely, states in which the primary government deficit is high, either because government expenditures are high or productivity is low, making $\text{cov}(\mathcal{R}_\tau, PV(\mathcal{X}_\tau)) > 0$. Therefore, equation (47) implies that an optimal long-run level of risk-free debt is negative, that is, the planner accumulates assets. Furthermore, since aggregate consumption growth is not volatile, at least in U.S. data, $\text{var}(\mathcal{R}_\tau)$ would be low in most U.S. calibrations, implying that the long-run asset level should be quite high (see also Example 2 in Section III.B). This provides intuition for some of the numerical findings in Aiyagari et al. (2002) and some subsequent contributions.

We can apply insights from Section III.B to situations in which the planner manages a portfolio of K securities. A version of the planner's Bellman equation (43), modified to have effective total assets to be the state variable, extends along the lines in equation (23). In the interior, a martingale equation restricts every security, namely,

$$V'\left(\tilde{\mathcal{B}}_t, s_t\right) = \mathbb{E}_t \mathcal{R}^k_{t+1} V'\left(\tilde{\mathcal{B}}_{t+1}, s_{t+1}\right) = \mathbb{E}_t V'\left(\tilde{\mathcal{B}}_{t+1}, s_{t+1}\right) + \beta \text{cov}_t(\mathcal{R}^k_{t+1}, V'(\tilde{\mathcal{B}}_{t+1}, s_{t+1})), \tag{48}$$

where $\mathcal{R}^k_{t+1}$ is the effective return on security k. Equation (48) implies equation (34) of Farhi (2010) that describes CCAPM Euler equations. Now let $\mathcal{R}^k_\tau$ be the effective returns on asset k evaluated at tax rate τ, and let $\boldsymbol{\mathcal{R}}_\tau = [\mathcal{R}^1_\tau ... \mathcal{R}^K_\tau]$ be a matrix of these returns. Combining the insights from this section and Lemma 4, it follows that the risk-minimizing portfolio can be approximated, up to the order $\mathcal{O}(\sigma, 1-\beta)$, by

$$-\mathbb{C}[\boldsymbol{\mathcal{R}}_{\tau(0)}, \boldsymbol{\mathcal{R}}_{\tau(0)}]^{-1}\mathbb{C}[\boldsymbol{\mathcal{R}}_{\tau(0)}, PV(\mathcal{X}_{\tau(0)})]. \tag{49}$$

IV. A Quantitative Example

We now study an economy with a risk-averse representative consumer together with g, p, θ processes calibrated to match stylized U.S. business cycle facts during the post-World War II period. We use this calibrated economy to verify the accuracy of our approximations for the ergodic behavior of government debt, the tax rate, and tax collections under an optimal plan. Specifically, we compare the means, variances, and convergence speeds using the expressions in Proposition 7, equation (49) governing the risk-minimizing portfolio, and related extensions of expressions for other moments reported in Proposition 6.

We set utility function parameters α, γ, β equal to 1, 2, 0.98. We begin by assuming that households and the government trade a single one-period security and parameterize a stochastic process for (θ_t, p_t, g_t) in terms of the following AR(1) specifications:

$$\begin{aligned}
\ln \theta_t &= \rho_\theta \ln \theta_{t-1} + \sigma_\theta \epsilon_{\theta,t} \\
\ln g_t &= \ln \bar{g} + \chi_g \epsilon_{\theta,t} + \sigma_g \epsilon_{g,t} \\
\ln p_t &= \chi_p \epsilon_{\theta,t} + \sigma_p \epsilon_{p,t},
\end{aligned}$$

where $\epsilon_{\theta,t}$, $\epsilon_{g,t}$, and $\epsilon_{p,t}$ are i.i.d. standard normal random variables.

Our parameterizations of productivity and government expenditures are standard, but our calibration of asset payoffs is less common. The literature typically assumes that the real payoff on government debt is risk-free and calculates returns on that asset from a marginal utility of a representative consumer within a neoclassical growth model. This approach unfortunately implies asset returns that are not consistent with observed returns on government debt. That deficiency matters for us because our formulas assert that the variance and covariance of returns on government debt are important determinants of optimal debt management. We set parameters of the stochastic process of payoffs p_t to assure that the return on the government's portfolio matches the return on the security held or issued by the government in our model.[22]

Table II documents our calibration targets for parameters $(\bar{g}, \rho_\theta, \chi_g, \chi_p, \sigma_\theta, \sigma_g, \sigma_p)$ in terms of moments of output, government expenditures, and bond returns. We use time series for these variables for 1947–2014 at annual frequencies. Except for returns, we took logarithms of all variables and then Hodrick-Prescott prefiltered them, using a smoothing parameter equal to 6.25. For

22. The finance literature argues for a stochastic discount factor that is sufficiently volatile and has a predictable component to account for premia and volatility of returns. In Online Appendix II we show that that our model with payoff shocks is essentially equivalent to a model with discount factor shocks. The stochastic process for discount factor shocks can be reverse engineered such that one can replicate the desired asset pricing moments. However, in line with the payoff shock framework used throughout the article, we chose to calibrate p_t directly.

TABLE II
PARAMETERS AND TARGETED MOMENTS IN THE COMPETITIVE EQUILIBRIUM WITH FITTED U.S. TAX POLICIES

Parameter	Value	Moment	Model	Data
Log output				
σ_θ	0.02	Std. dev.	1.7%	1.6%
ρ_θ	0.35	Auto corr	0.23	0.23
Returns				
σ_p	0.05	Std. dev.	5.2%	5.1%
χ_p	0.25	Corr with $\log y_t$	−0.008	−0.004
Log expenditures				
$\bar{g}$	0.26	Mean $\frac{g_t}{y_t}$	26%	26%
σ_g	0.02	Std. dev.	2.6%	2.6%
χ_g	−0.2	Corr with $\log y_t$	−0.14	−0.15

output y_t and government expenditures g_t, we use Bureau of Economic Analysis data for aggregate real labor earnings and federal government consumption expenditures plus transfer payments.[23] We measure B_t as the real market value of gross federal debt series published by the Federal Reserve Bank of Dallas.[24]

We propose two measures of returns on government debt. As a baseline, we impute real returns R_t using data on the real federal primary deficit[25] X_t and market value of government debt B_t. The observed duration of government debt has been approximately constant, allowing us to write the government budget constraint as

$$(p_t + q_t)\mathrm{B}_{t-1} = q_t\mathrm{B}_t - X_t. \tag{50}$$

Multiply and divide the first term by q_{t-1} and use the fact that the holding period return for long-term debt is $R_t = \frac{q_t+p_t}{q_{t-1}}$ to rewrite

23. Since in our model we abstract from capital, our measure of output y is aggregate labor earnings. Results remain essentially unchanged if we use GDP per capita instead.

24. Calculation of this series takes into account outstanding marketable and nonmarketable debt of different maturities issued by the Treasury and uses current market prices to convert par value to market value.

25. We measure this as government expenditure, that is, federal consumption and transfer payments, minus total federal tax receipts, both from the Bureau of Economic Analysis.

TABLE III
OLS ESTIMATES FOR TAX RULE

Parameter	Value
$\bar{\tau}$	0.25 (0.021)
$\rho_{\tau_{-}}$	0.19 (0.14)
$\rho_{y}, \rho_{y_{-}}$	0.09 (0.08), 0.21 (0.08)
$\rho_{g}, \rho_{g_{-}}$	0.11 (0.06), 0.11 (0.06)
$\rho_{R}, \rho_{R_{-}}$	0.04 (0.03), −0.02 (0.03)
$\rho_{B_{-}}$	0.02, (0.05)

Note. The numbers in brackets are standard errors.

equation (50) as

$$R_t = \frac{B_t - X_t}{B_{t-1}}, \tag{51}$$

where $B_t = q_t \mathrm{B}_t$ is the observed market value of government debt. The average annual return in our sample is about 5% and its standard deviation is 5%. As an alternative measure, we also calibrate payoffs to match the moments of the real one year U.S. Treasury yield, obtained from release H.15 of the Board of Governors of the Federal Reserve System. The average return in our sample is 2.0% with a standard deviation of 2.6%. The main difference between the two return measures comes from the fact that capital gains from revaluations of long-term debt are captured in imputed returns but not in one-year Treasury yields.

Returns in our model are endogenous and depend both on parameters and on government policy $\{\tau_t, B_t\}_t$. We assume that the tax rate conformed to the rule

$$\tau_t = (1-\rho_\tau)\bar{\tau} + \rho_\tau \tau_{t-1} + \rho_Y \log y_t + \rho_{Y\_} \log y_{t-1} + \rho_g g_t + \rho_{g\_} g_{t-1}$$
$$+ \rho_R R_t + \rho_{R\_} R_{t-1} + \rho_{B\_} \log B_{t-1}, \tag{52}$$

whose coefficients we estimated with an OLS regression using our series on output, expenditure, returns, debt, and an average marginal income tax rate τ_t obtained from Barro and Redlick (2011). Our specification (52) is flexible enough to capture how tax rates are persistent and how they adjust to movements in government expenditures, returns, and the level of government debt. We report estimated coefficients of equation (52) in Table III and the in-sample fit in Figure II. Given our estimated tax rule, we set debt $\{B_t\}_t$ to satisfy the government's budget constraint.

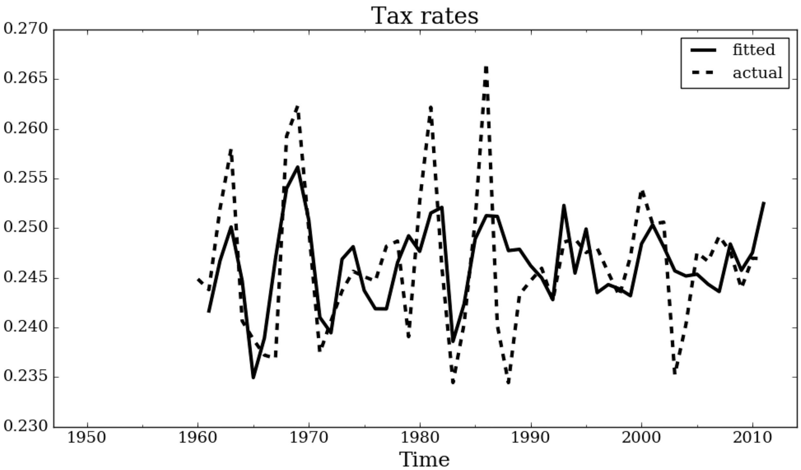

FIGURE II

Fitted Debt versus (H.P. Filtered) Average Marginal Tax Rates

Online Appendix III provides details about how we compute a competitive equilibrium given government policy $\{\tau_t,\ B_t\}_t$. Table II summarizes parameter values and the fit of a competitive equilibrium outcomes to U.S. data.

Using this calibration, we compute a global approximation to the Ramsey allocation using time iteration on the Euler equations of the planning problem.[26] Online Appendix III reports details about the numerical procedure. In Table IV, we compare predictions of our quadratic approximations about the behavior of government debt and tax revenues to those obtained by using a more accurate global numerical procedure. Given our assumption of logarithmic utility, effective debt and returns are simply $\mathcal{B}_t = \frac{B_t}{c_t}$ and $\mathcal{R}_t = \frac{R_t c_t}{c_{t-1}}$. Following Proposition 7, we use equation (47) evaluated at $\tau(0)$ to calculate the risk-minimizing level of debt and var$(\mathcal{R}_{\tau(0)})$ to compute the speed of mean reversion. We similarly use equations in Proposition 6, now written in terms of

26. In our case the optimal policies for effective debt and taxes can be cast as functions of the derivative of the value function of the planner. An accurate approximation of the Euler equation provides much more precise information of the slope of the value function than would an accurate approximation of its level. See McGrattan (1996) and Judd (1998) for more details on Euler equation-based projection methods and value function iteration methods.

effective units, to compute the ergodic variance of effective debt and moments of tax rates $Z_t = \tau_t(1-\tau_t)^{\frac{1}{\gamma}}$.

We computed the ergodic distribution by simulating policies computed using the global approximation method. The first two columns in Table IV show that for the baseline calibration, our expressions for the ergodic distribution of debt and tax revenues approximate well those obtained from the simulations. As an illustration of how the approximations do away from the ergodic distribution, we plot $\mathbb{E}_0\mathcal{B}_t$ using

$$\mathbb{E}_0[\mathcal{B}_t - \mathcal{B}^*] \approx (\mathcal{B}_t - \mathcal{B}^*)\left(\frac{1}{1+\beta^2 \text{var}(\mathcal{R}_{\tau(0)})}\right)^t, \tag{53}$$

and compare it to the mean path of length 15,000 periods constructed using 10,000 simulations under policies computed using the more accurate global methods. Figure III indicates that equation (53) gives a very accurate approximation for the entire path and not just its long-run target level of effective debt.

An insight of Proposition 7 is that covariances and variances are not very sensitive to the policies under which they are evaluated. Therefore, one should expect that the calculation of these variances in the data, generated by the actual rather than an optimal policy, produce reliable estimates of the optimal long-run debt level. We verify this as follows. Consider a simple first-order VAR

$$\begin{bmatrix} \mathcal{X}_t \\ \log y_t \end{bmatrix} = A \begin{bmatrix} \mathcal{X}_{t-1} \\ \log y_{t-1} \end{bmatrix} + \Sigma \begin{bmatrix} \epsilon_{\mathcal{X},t} \\ \epsilon_{y,t} \end{bmatrix}.$$

Let $[a_{\mathcal{X}} \;\; a_y]$ be the first row of the matrix $[I - \beta A]^{-1}$. Then the expected present value of the primary government surplus conditional on $(\mathcal{X}_t, \log y_t)$ is

$$PV\left(\mathcal{X}; (\mathcal{X}_t, \log y_t)\right) = \begin{bmatrix} a_{\mathcal{X}} & a_y \end{bmatrix} \begin{bmatrix} \mathcal{X}_t \\ \log y_t \end{bmatrix},$$

and an appropriate estimate of the target level of debt is

$$\mathcal{B}^* = -a_y \frac{\text{cov}(\mathcal{R}_t, \log y_t)}{\text{var}(\mathcal{R}_t)} - a_{\mathcal{X}} \frac{\text{cov}(\mathcal{R}_t, \mathcal{X}_t)}{\text{var}(\mathcal{R}_t)}.$$

We use our time series for returns, consumption, output, and the primary government deficit to construct time series of $\mathcal{R}_t$ and $\mathcal{X}_t$.

TABLE IV
ERGODIC MOMENTS FOR EFFECTIVE DEBT AND TAX REVENUES

Moments	Baseline			1 yr. yields		Risk-free bond	
	Global solution	Quadratic approx.	VAR	Global solution	Quadratic approx.	Global solution	Quadratic approx.
Effective debt, $\mathcal{B}_t$							
Mean	−7%	−7%	−6%	−24%	−23%	−42%	−42%
Half-life (years)	237	244	249	655	678	1,244	1,299
Std.	18%	20%	—	25%	33%	18%	46%
Tax rates, Z_t							
Mean	20%	20%	—	20%	20%	20%	20%
Half-life (years)	263	244	—	667	678	1,234	1,299
Std.	0.2%	0.4%	—	0.3%	0.7%	0.2%	0.9%

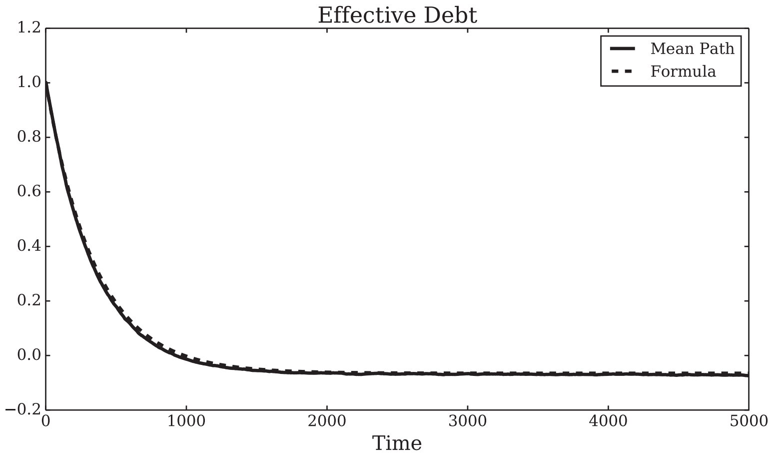

FIGURE III

Conditional Mean Paths of Effective Debt

The solid line is the conditional mean path for effective debt, $\mathbb{E}_0\mathcal{B}_t$ after averaging across 10,000 simulated paths. The dashed line is computed using the equation (53).

TABLE V

VAR ESTIMATES

Parameter	Value
α_y	−0.83
α_χ	0.50
$\frac{\text{cov}(\mathcal{R},\log y)}{\text{var}(\mathcal{R})}$	0.063
$\frac{\text{cov}(\mathcal{R},\chi)}{\text{var}(\mathcal{R})}$	−0.006
$\text{var}(\mathcal{R})$	0.003

Table V presents the estimated coefficients and moments that we then use to estimate both the target level of effective debt and the speed of the mean reversion reported in the column titled VAR in Table IV.

The findings in Table IV convey that at our baseline calibration the long-run effective debt is close to 0, that the convergence is slow (half-life of 237 years), that government debt has large fluctuations (the standard deviation is 18%), while there are small movements in the tax rate represented by Z_t (the standard deviation is 0.2%). A key empirical fact that drives these results is that a substantial component of fluctuations in returns is uncorrelated with fundamentals. That makes holding large positions frustrate the hedging motive and drives the optimal plan toward low assets.

In our baseline, we chose the payoff process to match imputed returns on the total debt portfolio traded by the government. To check robustness of our results we show that our approximation procedure continues to work when we instead measure the returns using the one-year U.S. Treasury yield or assume that the real debt traded by the government is risk-free. In Table IV, the columns "1 yr. yield" and "Risk-free debt" report moments of the ergodic distribution for our calibrated economy in which we set $\chi_p = -0.10$, $\sigma_p = 0.02$ to match the standard deviation of one-year U.S. Treasury yields and the correlation of those yields with output, which are 2.6% and -0.20 in our sample, respectively, and then, alternatively, $\chi_p = \sigma_p = 0$ to obtain the risk-free payoff. These alternative assumptions about returns progressively weaken the orthogonal component. Consistent with our discussion of equation (47), the government holds a larger asset position (i.e., a negative debt) to exploit the stronger positive correlation of returns and deficits. Because the speed of mean reversion is inversely related to the volatility of returns, the half-life of debt increases from 237 years in the baseline calibration to 655 years for the calibration with one-year yields and increases further to 1,244 years for the risk-free debt. In all of these settings, our simple formulas capture the comparative outcomes extremely well.

We now extend our analysis to allow the government to trade multiple assets. We pursue two aims with this extension. First, we want to evaluate the accuracy of approximations provided by equation (49). Second, we want to highlight additional insights about optimal government portfolio management and to reexamine an argument of Lucas and Zeldes (2009) that it is optimal for a government to take a positive position in a risky security that pays a risk premium. Although our problem has some features in common with the problem solved by Merton (1969), there are two critical differences: our problem is posed within a general equilibrium in which a Ramsey planner takes into account how its government actions affect asset returns, and the Ramsey planner is benevolent.

We fix parameters as described above except that now we assume that the government trades two securities. One is a riskless

TABLE VI
ERGODIC PORTFOLIO USING GLOBAL SOLUTION AND FORMULA (49)

Portfolio holdings	Global solution	Quadratic approx.
Risk-free bond	−42.40%	−43.00%
Risky asset	−0.05%	0.06%

real bond; the logarithm of the payoff on the other security is described by $\ln p_t = \chi_p \epsilon_{\theta,t} + \sigma_p \epsilon_{p,t}$, where χ_p, σ_p are now calibrated to match the correlation of dividends on the S&P500 with output and the standard deviation of these dividends, which for our sample take the values 0.30 and 4.5%. Making the payoffs positively comove with total factor productivity (TFP) results in higher expected holding period returns relative to the risk-free rate.[27] We set the initial debt at 130% of output.

From Table VI we see that the formula in equation (49) accurately describes the long-run portfolio in addition to the speed of mean reversion and standard deviation of total assets. In the long run, the optimal plan has negative debt invested almost solely in the risk-free asset. Initially, when it is indebted, the government shorts the stock market. Although the initial short position in the stock market exposes the government to the orthogonal component $\epsilon_{p,t}$ in the payoff, temporarily it provides a good hedge by delivering higher returns in times of low TFP. Eventually, the government uses only the risk free bond to hedge. The dynamics of portfolio rebalancing are consistent with Example 3 from Section III.B. The two subplots in Figure IV show the marginal utility adjusted government positions in the risk-free security (top) and in the risky security (bottom). The figure plots the mean path of the portfolio positions computed using the global approximation described above (solid lines) and using extensions of our formulas from Proposition 5 to risk aversion[28] (dashed lines). As with the

27. Given our assumption of isoelastic preferences, we cannot match the risk premium quantitatively, but we conjecture that our approach extends to Epstein-Zin preferences and richer environments with more realistic implications for asset pricing behavior such as one considered by Albuquerque et al. (2016). We leave this extension to future work.

28. Extending the formulas amount to replacing $\mathbf{R}$ with $\boldsymbol{\mathcal{R}}_{\tau(0)}$ returns and g with $PV(\mathcal{X}_{\tau(0)})$.

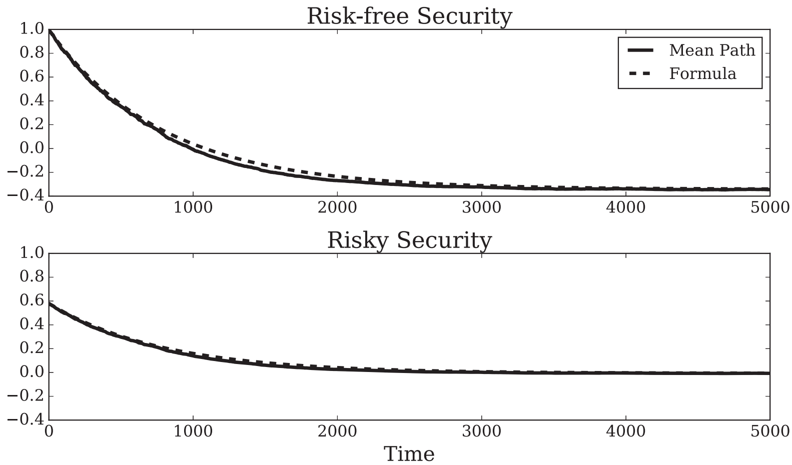

FIGURE IV

Conditional Mean Paths of Portfolio Holdings

Marginal utility weighted holdings of the risk-free bond (top) and "stock market" security (bottom). Solid lines represent paths computed using the global methods, and dashed line represent our approximations. Positive values imply that the government is shorting the security; thus, the government is initially in debt holding negative positions in both securities.

single asset case, our approximations capture the convergence to as well as the level of the ergodic mean.

Our quantitative analysis confirms the optimality of a portfolio management rule based on the variance-minimization principle outlined in Section III.B and cautions against following recommendations that a government should on the margin invest in assets that pay a risk premium. In our economy, the Ramsey planner shares households' aversion to consumption risk, an aversion that in general equilibrium requires a return premium to compensate for bearing risk. The Ramsey planner finds it optimal to invest in such assets only in so far as doing so helps reduce the total riskiness of gross government expenditures.

V. CONCLUDING REMARKS

This article characterizes optimal debt management and flat rate taxation in a fairly general incomplete markets model. We express dynamic hedging motives in a terms of a fiscal risk minimization problem. We present simple formulas for the mean, variance, and speed of convergence to an ergodic distribution of government debt. We analyze some extensions of our basic environment, an endeavor we pursue in Bhandari et al. (2017b), where we study economies in which persistent differences in skills unleash social motives for redistribution and social insurance. The analysis here sets the stage for such extensions—partly by providing appropriate tools for approximating equilibria well and for formulating Ramsey problems in mathematically convenient ways, and partly by isolating transcendent forces that ultimately determine transient and long-run dynamics of government debt and taxes in richer settings. For example, appropriately adjusted fiscal risk minimization problems continue to shape the mean of an ergodic distribution of government debt, while the hedging costs of being away from that fiscal risk-minimizing debt level shape speeds of convergence. Another extension, Bhandari et al. (2017a), uses the empirical properties of returns across maturities to compute an optimal maturity structure of government debt.

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Supplementary Material

An Online Appendix for this article can be found at *The Quarterly Journal of Economics* online.

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