RANDOM GRAPHS AND KEISLER'S ORDER

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Abstract. We survey the complexity of various structures similar to the random graph from the perspective of Keisler's order. The random graph is an unstable theory, and is in the minimal unstable class for Keisler's order. However, other structures defined in similar ways have drastically different classifications. For instance, we can require that there be no "triangles," and this defines a structure in the maximum class. We can also generalize the definition of the random graph to form random hypergraphs. Structures based on the random hypergraphs form an infinite descending chain, which gives a proof that Keisler's order has infinitely many classes.

1. Introduction

Our goal in this paper is to introduce the random graph and several variants, and to analyze them from the perspective of Keisler's order. We assume familiarity with the basic notions of model theory, including (regular) ultrafilters, ultraproducts and saturation, but no particular familiarity with Keisler's order. Basic information on these topics can be found in [1], and many more details are in [7].

First, we give the definition of the random graph. For the purpose of this paper, all graphs are simple — that is, they have neither loops nor multiple edges. Furthermore, all our graphs will be undirected.

Definition 1.1. The *random graph*, which we write Rg, is a countable model of the theory of graphs with the following property: For any $a_1, \dots, a_n, b_1, \dots, b_m \in G$, there exists $c \in G$ which has an edge to each a_i and none of the b_j, as long as $a_i \neq b_j$ for any i, j.

Before going any further, let's justify our use of the definite article:

Lemma 1.2. *The random graph exists and is unique up to isomorphism.*

Proof. Using induction, we will construct a graph G with the desired property. At stage 1, add a single node to the graph. At stage n, we have constructed some finite graph G_n. Since the graph is finite, there are only finitely many pairs of tuples $\langle a_1, \dots, a_n\rangle, \langle b_1, \dots, b_m\rangle$, with $a_i, b_j \in G_n$, that we need to worry about. Therefore, for each such tuple, we can simply add a new node to the graph which connects to the others in the way we want. The finite graph constructed in this way becomes G_n. Let $G = \cup_{n\in\mathbb{N}} G_n$. Then G has the property described above, since any finite tuple will show up at some stage G_n and then be satisfied at stage G_{n+1}.

To show uniqueness, suppose G, H both have the property given above. Applying countability, enumerate the graphs as $G = \{g_n\}$, $H = \{h_n\}$. Using a back-and-forth argument, we will construct an isomorphism between them. At stage n, let

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f_n be the partial isomorphism constructed so far, and let A_n be its domain. Our goal is to extend f_n to include g_n, assuming g_n is not already in A_n. To do this, note that g_n has edges to some g_i with $i \leq n$ and not to others. If we can find an element $c \in H$ which has the same pattern of edges with respect to the elements of $f_n(A_n)$, we can define $f_{n+1}(g_n) = c$. However, the existence of such an element c is an immediate consequence of the random graph property (in particular, since the map constructed so far is injective, we will not be demanding that c both connect and not connect to some element).

We can repeat the same procedure for h_n, picking some $d \in G$ and setting $f_{n+1}(d) = h_n$. In this way, we build bijective isomorphisms $f_n \colon A_n \to f(A_n)$ for every n, such that $g_n \in A_{n+1}$, $h_n \in f_{n+1}(A_{n+1})$. It follows that $\cup_{n\in\mathbb{N}} f_n \colon G \to H$ is a bijective isomorphism of graphs, completing the proof. □

Note that by effectively the same procedure given in the proof, we can embed any at-most countable graph as an induced subgraph of Rg. This shows that Rg is $\aleph_1$-*universal.*

There is an additional perspective on Rg which is worth mentioning, especially since it gives the structure its name. For any fixed $p \in (0, 1)$, we can imagine building successively larger structures in the following way: at each stage add a new node, and for each existing node, connect it to the new node with probability p. The finite structures constructed in this way are not unique. However, if we take the union of the stages in some specific construction, the result, with probability 1, will be the random graph.

2. Keisler's Order

The question we now want to address is: how "complex" is the structure Rg? One answer is provided by Keisler's order, which we now define.

Definition 2.1. Let M, M_1, M_2 be models. We say "D saturates M", for D a regular ultrafilter on λ, if M^λ/D is λ^+-saturated We write $M_1 \trianglelefteq M_2$ if, for any regular D, if D saturates M_2 then D saturates M_1.

Although Keisler's order is initially defined as a preorder on models, in fact it depends only on the theory satisfied by the model. Accordingly, we will often say that D saturates T, for T a complete countable first-order theory.

Lemma 2.2. *Let M, N be models of the complete, countable first-order theory T. Then for any D, D saturates M if and only if it saturates N.*

Proof. See Theorem 2.1 of [4]. □

Intuitively, the reason that Keisler's order measures complexity is because a more complicated structure allows encoding of more sensitive types, making it harder to satisfy all types in an ultrapower, while a simpler structure will tend to have relatively straightforward types. This will hopefully become clearer through our discussion of the random graph.

Keisler's order has a minimal class, which contains for instance the theory ACF_p of algebraically closed fields of a fixed characteristic p. It turns out that the union of the first two classes is precisely the theories which are "stable" (see [7], section VI.5). For more details on the definition of instability see [7], theorems II.2.2 and II2.2.13. We will only use the following:

Definition 2.3. Let T be a (complete, countable) theory. The formula $\phi(x, y)$ in the language of T has the order property if, in any model $M \vDash T$, there exist $\{a_n\}$, such that for every k, $\{\phi(x, a_i) : i \leq k\} \cup \{\neg\phi(x, a_j) : j > k\}$ is consistent (notice that this does not depend on the model). The theory T has the order property if some formula ϕ has the order property.

Theorem 2.4. *A theory has the order property if and only if it's unstable.*

In the definition of the order property and elsewhere, parameters are consistently allowed to be tuples rather than single elements. We write 'y' rather than '$\bar{y}$' to avoid clutter.

Let T_{Rg} be the theory of the random graph, with R acting as the edge relation. We will show that the formula $\phi(x, y) = R(x, y)$ has the order property, which means T_{Rg} is not in the first two classes of Keisler's order. This is easy: pick any sequence $\{a_n\}$ of distinct elements from Rg. The type $\{R(x, a_i) : i \leq k\} \cup \{\neg R(x, a_j) : j > k\}$ is consistent if, and only if, every finite subset is. However, any finite subset of this type is trivially consistent by the defining property of the random graph. Thus, T_{Rg} is unstable.

In fact, by the exact same proof, we can divide parameters however we want: Given *any* $A \subset \mathbb{N}$, the type $\{R(x, a_i) : i \in A\} \cup \{\neg R(x, a_j) : j \notin A\}$ is consistent. This is very different from the case of a linear order: for instance, if we took our model to be $(\mathbb{N}, <)$ and set $a_n = n$, $\phi(x, y) = x < y$, then the order property holds. However, in the notation given above, the type will never be consistent unless A is entirely above its complement. These considerations suggest the following definitions:

Definition 2.5. Let T be a theory, and let $\phi(x, y)$ be a formula.

(1) ϕ has the independence property (IP) if there exist $a_i \in M$, for some $M \vDash T$, such that for any $A \subset \omega$, $\{\phi(x, a_i) : i \in A\} \cup \{\neg\phi(x, a_j) : j \notin A\}$ is consistent. Equivalently, ϕ has IP if there exist a_i such that $\{\phi(x, a_i) : i \in \sigma\} \cup \{\neg\phi(x, a_j) : j \in \tau\}$ is consistent for $\sigma, \tau \subset \omega$ finite and disjoint.
(2) ϕ has the strict order property (SOP) if there exist $a_i \in M$, for some $M \vDash T$, such that $\phi(x, a_i) \wedge \neg\phi(x, a_j)$ is consistent if and only if $i < j$.

A theory T has IP or SOP if some formula in the language of T has IP or SOP.

It turns out that these two properties jointly characterize all unstable theories.

Theorem 2.6 ([7], Theorem II.4.7.(1))**.** *T is unstable if and only if T either has the independence property or the strict order property.*

Furthermore, all theories with SOP are in the maximum class of Keisler's order, as proven in [7], theorem VI.4.3. A few cautionary notes are in order here. First, theories can have both IP and SOP, although a specific formula cannot. Second, although all theories with SOP are maximal, it need not be the case that a theory in the maximum class has SOP.

3. The Class of the Random Graph

By the argument we gave above, it's clear that T_{Rg} does not have SOP, so since the theory is unstable, it must have the independence property. We will show that T_{Rg} occupies the minimum class among unstable theories. Since theories with

SOP are maximal, it's enough to show T_{Rg} is minimal among theories with the independence property.

To complete the argument, we will need a few standard facts and definitions. First, the theory T_{Rg} has quantifier elimination. The proof of this is similar to the standard proof that $(\mathbb{Q}, <)$ admits quantifier elimination, and in any case is given in [2]. Next, we will need the following result:

Lemma 3.1. *Let D be a regular ultrafilter on λ, and let $A \subset M^\lambda/D$ be a set of size at most λ. Then there exist sets n_i for $i \in \lambda$, each a finite subset of M, such that $A \subset \prod n_i/D$.*

Proof. Pick some regularizing family $\{X_i : i \in \lambda\}$. This means that each $X_i \in D$, and that any infinite intersection of elements X_i is empty. In different language, this means that each $p \in \lambda$ is contained in only finitely many sets X_i. Fix some enumeration $A = \{a_i : i \in \lambda\}$, and stipulate that $a_i[j] \in n_j$ if and only if $j \in X_i$. Then each n_j is finite, and the projection of each a_i is contained in n_j on the large set $X_i \in D$, so A is contained in the product $\prod n_i/D$. $\square$

The final notion we will need is that of an internal set: $X \subset M^\lambda/D$ is internal if there exist X_i for $i \in \lambda$ such that $X = \prod X_i/D$.

Lemma 3.2. *Let D be a regular ultrafilter on λ. If, for any infinite model M and any two disjoint $A, B \subset N = M^\lambda/D$ each of size at most λ, there exists an internal set X which separates them, then D saturates T_{Rg}.*

Proof. Pick some (partial) type p of size at most λ. Since the theory of the random graph has quantifier elimination, we may assume the type under consideration is of the form $\{R(x,a) : a \in A\} \cup \{\neg R(x,b) : b \in B\}$, for some disjoint sets A, B such that $|A| + |B| \leq \lambda$. For each t, consider the sets $A[t] = \{a[t] : a \in A\}$, and $B[t] = \{b[t] : b \in B\}$. Since D is regular, all sets of size at most λ are covered by products of finite sets, so we may assume without loss of generality that $A[t], B[t]$ are finite for each choice of t. Applying the separation property of D, there exists some internal set X with $A \subset X$, $X \cap B = \emptyset$.

Applying the definition of an internal set, we can write $X = \prod X_i/D$ for some sets X_i. For each t, use the random graph property to pick an element c_t connecting to all of $A[t] \cap X_t$ and none of $B[t] \setminus X_t$: we can do this, because the two sets are finite and disjoint. Define $c = \langle c_t : t \in \lambda \rangle/D$. We claim that c realizes the type. Indeed, for any $a \in A$, $a[t] \in X_t$ on a large set, since $a \in X$. Whenever $a[t] \in X_t$, we know that $a[t]$ and c_t will have an edge, so this means that they have an edge on a large set, so a and c have an edge. The same argument goes for the elements of B, showing that the type is realized. $\square$

Lemma 3.3. *Suppose the model M has a formula ϕ with the independence property. Then any regular ultrafilter which saturates M must separate small sets in the sense given above, and hence must saturate T_{Rg}.*

Proof. By definition of the independence property, there exist $\langle a_i : i \in \omega \rangle$ where $\{\phi(x, a_i) : i \in \sigma\} \cup \{\neg\phi(x, a_i) : i \in \tau\}$ is consistent for any finite, disjoint $\sigma, \tau \subset \omega$. Define $\theta(x, y, z, w) = ((z = w) \implies \phi(x, y)) \wedge ((z \neq w) \implies \neg\phi(x, y))$. Pick (without loss of generality) any b not contained in the sequence $\langle a_i \rangle$. We define an array of elements a_{ij}, where $i \in \omega$ and $j = 0, 1$, as follows: $a_{n0} = (a_n, a_n, a_n)$, $a_{n1} = (a_n, a_n, b)$.

The formula $\theta(x, \bar{y})$, where $\bar{y} = (y, z, w)$ has the following property: For any finite subset $\sigma \subset \mathbb{N} \times \{0, 1\}$, the conjuction $\{\theta(x, \bar{a_{ij}}) : (i, j) \in \sigma\}$ is consistent if and only if it does not contain two elements a_{i0}, a_{i1}. This is because any such finite subset asserts finitely many ϕ-sentences, which will lead to contradiction if and only if some two parameters are pulled from the same row.

Now, let A, B be disjoint subsets of M of size at most λ. For each t, $A[t]$ and $B[t]$, which we may take to be finite, will have some pattern of intersection. At each t, map each element of $A[t]$ to some a_{i0} and each element of $B[t]$ to some a_{i1}, only mapping them to the same i if they're the same element. Now, for any finite subset of A and any finite subset of B, their projections are disjoint most of the time. It follows that the type $\{\phi(x, \tilde{a}) : a \in A\} \cup \} \phi(x, \tilde{b}) : b \in B\}$ is consistent, where $\tilde{a}$ is the equivalence class of the sequence of elements a_{i0} which we sent a to in each index model. Since this type is small and D is good for M, it must be realized by some element c.

What this means is that for any finite set of elements a and b, we have $\phi(c[t], \tilde{a}[t])$ and $\neg\phi(c[t], \tilde{b}[t])$ on a large set. Then, simply define $X_t = \{x : \phi(c[t], x)\}$. X is an internal set, and it follows immediately that X separates A and B. This completes the proof. $\square$

4. Generalizations and Extensions

In this final section, we explain some generalizations of the random graph and discuss their relationship to relatively recent results. In one direction, we could consider adding restrictions to the connections allowed in the graph.

Definition 4.1. The triangle-free random graph, $G_{3,2}$ is a countable random graph which satisfies the following property: for any $a_1, \ldots, a_n$, $b_1, \ldots, b_m$, there exists c which is connected to all the a_i and none of the b_j, as long as it doesn't force the existence of a complete subgraph on three vertices (a "triangle"). For any $n \geq 3$, the K_n-free random graph, $G_{n,2}$ is defined analogously. The theory $T_{n,2}$ is the complete theory of the model $G_{n,2}$.

Although the definition is superficially similar, it turns out that $T_{n,2}$ is significantly more complex from the perspective of Keisler's order. This follows from recent work on another property of models, which we introduce now.

Definition 4.2. A complete, countable theory T has SOP_2 for some formula $\phi(x, y)$ and some model $M \vDash T$, there exist parameters $\{a_\eta : \eta \in {}^{\omega>}2\}$, indexed by a binary tree, with the following property: for any distinct $\eta_1, \ldots, \eta_n \in {}^{\omega>}2$, $\{\phi(x, a_{\eta_1}), \ldots, \phi(x, a_{\eta_n})\}$ is consistent if and only if the parameters lie on the same branch.

We will prove that the triangle-free random graph has SOP_2 (the proof for the general K_n-free random graph is similar). The key fact used in the proof is the following, an analogue of the corresponding property for the random graph:

Lemma 4.3. *Suppose G is an at-most countable triangle-free graph. Then G embeds into $G_{3,2}$.*

Proof. Enumerate $G = \{g_n\}$: we will recursively construct an embedding $f : G \to G_{3,2}$. Suppose we have already embedded $g_1, \ldots, g_n$ into $G_{3,2}$. In order to extend the embedding to g_{n+1}, we need to find an element c, not yet chosen, which has the same pattern of edges with respect to $f(g_1), \ldots, f(g_n)$ that g_{n+1} has with respect to $g_1, \ldots, g_n$. This just amounts to requiring that c have an edge to certain elements and not to others. Furthermore, since G is triangle-free, none of the required edges force the existence of a triangle. Therefore, by the defining property of $G_{3,2}$, we can find the element we want. This shows that we can extend the embedding to all of G. □

Theorem 4.4. *The theory of the triangle-free (or K_n-free) random graph has SOP_2.*

Proof. Let $\phi(x, \bar{y}) = \phi(x, y, z) = R(x, y) \wedge R(x, z)$. To prove the theory of $G_{3,2}$ has SOP_2, we need to find a collection of pairs of elements, indexed by a binary tree, with the property given above. We define the following graph G: its distinct nodes are labeled a_η, b_η, for $\eta \in {}^{\omega>}2$. For any η, ρ, a_η and b_ρ have an edge if and only if η and ρ are incomparable (that is, on different branches), and these are the only edges. The graph G defined in this way is bipartite: there are no edges between elements a_η, a_ρ, nor between elements b_η, b_ρ. Since G is bipartite, in particular G is triangle-free. It follows by the lemma proved above that G embeds into $G_{3,2}$, so we may take $a_\eta, b_\eta \in G_{3,2}$.

Now, for each $\eta \in {}^{\omega>}2$, let the corresponding parameter be (a_η, b_η). We need to check this satisfies the definition of SOP_2. Pick any $\eta_1, \ldots, \eta_n$: first, suppose they lie on the same branch. Then all these elements are comparable, so by the definition of G given above, there are no edges between the elements $a_{\eta_1}, b_{\eta_1}, \ldots, a_{\eta_n}, b_{\eta_n}$. On the other hand, suppose η_i, η_j are incomparable: Then a_{η_i} connects to b_{η_j}. Therefore, if $R(x, a_{\eta_i})$ and $R(x, b_{\eta_j})$ both held, then the graph would have a triangle, and it follows that the conjunction is inconsistent. This completes the proof. □

It was recently proven in [5] that any theory with SOP_2 is maximum in Keisler's order. Therefore, we conclude that the K_n-free random graph is in the maximum class.

We also might try generalizing the random graph to hypergraphs. An n-hypergraph, for the purpose of this paper, is a structure whose edge relation R takes n arguments, is symmetric in its arguments, and which only allows edges between n distinct nodes. A complete n-hypergraph is one in which all edge relations which are allowed to hold do hold. Extending the notation used for the triangle-free random graph, for any n, k we can define the *k-free n-hypergraph*, $G_{k,n}$, as follows: it is the unique countable n-hypergraph such that for any $a_1, \ldots, a_p, b_1, \ldots, b_q$, each of which is a subset of $G_{k,n}$ of size exactly $n-1$, there exists some $c \in G_{k,n}$ which shares an edge with the tuple a_i for every i and not does not share an edge with b_j, unless the existence of such a c would force the existence of a complete subgraph of size k.

Based on the example of the triangle-free random graph, we might expect these structures to be maximal in Keisler's order. However, it turns out that the situation is much more interesting. Hrushovski proves in [3] that all these theories are "simple," falling into a category that generalizes the unstable theories. Furthermore, the following result was recently proved in [6]:

Theorem 4.5. *If $2 \leq k_1 < k_2$, then $T_{k_1,k_1+1} \ntrianglelefteq T_{k_2,k_2+1}$.*

In particular, this provides the first proof within ZFC that Keisler's order has infinitely many classes.

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