Higher Ramification in two-dimensional mod p Galois representations

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Let $K/\mathbb{Q}_p$ be a finite unramified extension of local fields of degree f, with residue field k. Let $G = G_K$ be its Galois group. Let $\rho\colon G_K \to V := \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a mod p Galois representation, always assumed to be continuous. For example, if E/K is an elliptic curve, and $E[p]$ is its group of p-torsion points defined over $\overline{K}$, we obtain an action of G_K on $E[p]$. Since $E[p] \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, this gives a map $G_K \to \mathrm{GL}_2(\mathbb{F}_p) \subset \mathrm{GL}_2(\overline{\mathbb{F}}_p)$.

In this paper, we describe how the higher ramification subgroups of G_K act on V. More precisely, we answer the following question: For which $u > 0$ is it possible that G^u acts nontrivially on V, but $G^{u+\varepsilon}$ acts trivially for all $\varepsilon > 0$? Except for a few sections, we almost entirely follow section 3 of [DDR16], but give more detail.

1 Semisimple representations

First, suppose that V is a semisimple G_K-representation. We will show that the higher ramification subgroups act trivially on V, regardless of the dimension of V. Let $P_K = G^{0^+} = \cup_{\varepsilon>0} G^{\varepsilon}$, where G^s denotes ramification groups in upper numbering. We call P_K the *wild inertia group.*

Theorem 1.1. *Let $\rho\colon G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be an irreducible representation. Then $\rho|_{P_K}$ is trivial.*

Proof. First, we show that $\rho|_{P_k}$ fixes a nonzero vector. Note that P_K is a pro-p group, in the sense that all its continuous finite quotients are p-groups. Indeed, if $A \subset P_k$ is a closed subgroup of finite index, then $A = \mathrm{Gal}(\overline{F}/F)$, where F is some finite extension of K^{tame}, and $P_K/A = \mathrm{Gal}(F/K^{\mathrm{tame}})$. But all finite extensions of K^{tame} are totally ramified with order a power of p, so this shows the claim.

We identify ρ with its restriction to P_K. Since ρ is continuous, and the target is discrete, ρ factors through some finite quotient P_K/A, and this finite quotient is a p-group as we discussed above. This likewise shows that ρ lands in $\mathrm{GL}_n(\mathbb{F}_{p^m})$ for some m. It follows by the orbit-stabilizer formula that all the orbits of the P_K-action on $(\mathbb{F}_{p^m})^2$ have order a power of p. Also, the orbit of the point 0 is just the point 0 itself. Since the space being acted on has cardinality p^{2m}, it follows by a counting argument that there must be some $v \neq 0$ fixed by the action of P_K.

We claim that the subspace V^{P_K} of points fixed by P_K is G_K-stable. Indeed, recall that $P_K \subset G_K$ is normal. For any $x \in V^{P_K}$, $p \in P_K, g \in G_K$, we have $p(g(x)) = g(p'(x)) = g(x)$ for some $p' \in P_K$, so $g(x) \in V^{P_K}$. It follows that V^{P_K} is a nontrivial G_K-subrepresentation. Since V was assumed irreducible, this shows that $V^{P_K} = V$, so P_K acts trivially on V. $\square$

It follows that for any $u > 0$, G_K^u acts trivially on any irreducible G_K-representation, which clearly shows that the same holds for any semisimple G_K-representation. This completely answers our question in the case where V is semisimple.

Returning to the dimension two case, suppose that V is not semisimple. It follows in particular that there is some one-dimensional G_K-stable subspace, on which G_K acts via a character χ_1. The quotient is likewise a one-dimensional representation, so we have an exact sequence of G_K-modules:

$$1 \to \overline{\mathbb{F}}_p(\chi_1) \to V \to \overline{\mathbb{F}}_p(\chi_2) \to 1$$

Consider the representation $W := V \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p(\chi_2^{-1})$, where we "twist" to eliminate the second character. We claim that understanding higher ramification for this representation is the same as understanding it for the original one. Indeed, since $\overline{\mathbb{F}}_p(\chi_2^{-1})$ is an irreducible G_K-representation, for any $u > 0$, any $g \in G_K^u$ acts trivially on $\overline{\mathbb{F}}_p(\chi_2^{-1})$. Therefore, any such g acts in the same way on $V \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p(\chi_2^{-1})$ as it does on $V \otimes_{\overline{\mathbb{F}}_p} \overline{\mathbb{F}}_p = V$, which proves our claim. Replacing V with the representation W if necessary, we may assume V fits into an exact sequence of the following form:

$$1 \to \overline{\mathbb{F}}_p(\chi) \to V \to \overline{\mathbb{F}}_p \to 1$$

Picking an appropriate basis, the action of G_K has the following form:

$$\rho(g) = \begin{pmatrix} \chi(g) & \alpha(g) \\ 0 & 1 \end{pmatrix},$$

where $\alpha\colon G_K \to \overline{\mathbb{F}}_p$ is some continuous function. Since ρ is a homomorphism, by multiplying matrices we see that $\alpha(gh) = \alpha(h)\chi(g) + \alpha(g)$. This means that α is a *cocycle* for χ: $\alpha \in Z^1(G_K, \overline{\mathbb{F}}_p(\chi))$, and conversely, any cocycle α will give a well-defined representation. This suggests that we should focus on understanding $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$, and leads us to define the following filtration:

Definition 1.2. For any $s \in \mathbb{R}$, we define:

1. $\mathrm{Fil}^s(H^1(G_K, \overline{\mathbb{F}}_p(\chi))) = \bigcap_{u > s-1} \ker\left[H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \to H^1(G_k^u, \overline{\mathbb{F}}_p(\chi))\right]$.
2. $\mathrm{Fil}^{<s}(H^1(G_K, \overline{\mathbb{F}}_p(\chi))) = \ker\left[H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \to H^1(G_K^{s-1}, \overline{\mathbb{F}}_p(\chi))\right]$

Thus, Fil^s represents the cocycles which are trivial restricted to G^{t-1} for any $t > s$, and $\mathrm{Fil}^{<s}$ represents the cocycles which are trivial on G^{s-1}. We claim that:

$$\mathrm{Fil}^{<s} = \bigcup_{t<s} \mathrm{Fil}^t$$

The inclusion $\cup_{t<s} \mathrm{Fil}^t \subset \mathrm{Fil}^{<s}$ is trivial. Conversely, pick some cocycle z and suppose $z \in \mathrm{Fil}^{<s}$. By continuity of z, $\ker(z)$ is open, and our assumption tells us that $G_K^{s-1} \subset \ker(z)$. It's clear that:

$$G_K^{s-1} = \bigcup_{n \in \mathbb{N}} G_K^{s-1-1/n}$$

Each set $A_n = G_K^{s-1-1/n}$ is closed, as is $(\ker(z))^c$. Since G_K is compact, the sets $B_n := A_n \cap (\ker(z))^c$ form a decreasing sequence of compact sets, and their intersection is $G_K^{s-1} \cap (\ker(z))^c = \emptyset$. It follows that at least one of the sets B_n must be nonzero, which shows that $z \in \cup_{t<s} \mathrm{Fil}^t$.

Note also that if $s < 0$ then $\mathrm{Fil}^s = 0$. Indeed, in this case Fil^s is contained in the kernel of the restriction map $H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \to H^1(G_K^{-1}, \overline{\mathbb{F}}_p(\chi))$. But $G_K^{-1} = G_K$, so this kernel is trivial.

We showed above that $\chi|_{P_k}$ is trivial. It follows that the restriction of any cocycle to P_K is a homomorphism, and that the restriction of any coboundary is trivial. Thus if $s > 1$ and $c \in H^1(G_K, \overline{\mathbb{F}}_p(\chi))$, then for any z representing c, $c \in \mathrm{Fil}^s$ if and only if $z(G_K^u) = 0$ for all $u > s-1$.

Furthermore, if α is a cocycle corresponding to the representation ρ (the choice of α depends on the basis), then we claim G_K^u acts trivially via ρ if and only if $\alpha|_{G_K^u}$ is trivial. Indeed, we showed that $\chi|_{P_K}$ is trivial, which shows that χ is trivial restricted to G_K^u, since $u > 0$. Considering the matrix for $\rho(g)$, it's clear that an element $g \in G_K^u$ acts trivially if and only if $\alpha(g) = 0$.

Say that ρ has a jump at $u > 0$ if G_K^u act nontrivially, but $G_K^{u+\varepsilon}$ acts trivially for all $\varepsilon > 0$. It follows that there exists some ρ with a jump at $u > 0$ exactly when $\mathrm{Fil}^{u+1} \neq \mathrm{Fil}^{<u+1}$. Define:

$$\mathrm{gr}^s = \mathrm{Fil}^s / \mathrm{Fil}^{<s}, d_s = \dim(\mathrm{gr}^s)$$

Then there exists ρ with a jump at u if and only if $d_{u+1} > 0$. It remains to determine where these jumps can occur.

2 Local Class Field Theory

In this section, we state some results which we will need to complete the argument.

Theorem 2.1 (Local class field theory)**.** *1. For any local field K, there is a homomorphism $\mathrm{Art}_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$, uniquely characterized by:*

- *(a) If π is a uniformizer of K, then $\mathrm{Art}_K(\pi)|_{K^{\mathrm{ur}}} = \mathrm{Frob}_K$.*
- *(b) If K'/K is a finite abelian extension, then $\mathrm{Art}_K(N_{K'/K}(K'^\times))|_{K'} = \mathrm{id}$.*

Moreover, Art_K is an isomorphism onto the Weil group W_K^{ab}.

2. If K'/K is a finite separable extension, then $\mathrm{Art}_{K'}(x)|_{K^{\mathrm{ab}}} = \mathrm{Art}_K(N_{K'/K}(x))$ for all $x \in K'^\times$, and Art_K induces an isomorphism $K^\times/N_{K'/K}(K'^\times) \xrightarrow{\cong} \mathrm{Gal}((K' \cap K^{\mathrm{ab}})/K)$.

Proof. See Theorem A of [Yos08]. □

We will also need the following compatibility result:

Theorem 2.2. *The map Art_K identifies the filtration of $K^\times$ by the unit groups U^s with the filtration of W_K^{ab} by the ramification groups $(G_K^{\mathrm{ab}})^s \cap W_K^{\mathrm{ab}}$.*

Proof. See [Ser79], page 228, corollary 3 to theorem 1 of chapter XV. □

We will need a few important facts about Galois cohomology, especially of local fields.

Proposition 2.3 (Inflation-restriction exact sequence)**.** *Let $H \trianglelefteq G_K$ be a closed normal subgroup, and let A be a G_K-module. Then we have an exact sequence:*

$$0 \to H^1(G_K/H, A^H) \to H^1(G_K, A) \to H^1(H, A)^{G_K/H} \to H^2(G_K/H, A^H) \to H^2(G_K, A)$$

Proof. See [Ser02], section I.2.6. □

Proposition 2.4 (Tate local duality)**.** *Let A be a finite G_K-module, and let A' denote* $\mathrm{Hom}(A, G_m)$*, where G_m is the G_K-module $\overline{K}^{\times}$. We have:*

$$H^2(G_K, A) \cong H^0(G_K, A')$$

Proof. See [Ser02], section II.5.2, theorem 2. □

Proposition 2.5. *Let A be a finite G_K-module, and define the Euler characteristic to be:*

$$EC(A) = |H^0(G_K, A)||H^1(G_K, A)|^{-1}|H^2(G_K, A)|$$

Then we have $EC(A) = p^{-[K:\mathbb{Q}_p]v_p(|A|)}$.

Proof. See [Ser02], section II.5.7, theorem 5. □

Proposition 2.6 (Unramified cohomology)**.** *Let A be a finite G_K-module such that I_K acts trivially on A. Then $|H^1(G_K/I_K, A)| = |H^0(G_K, A)|$. In particular, if B is* any *Galois module, then:*

$$|H^1(G_K/I_K, A^{I_K})| = |H^0(G_K, A)|$$

Proof. For the first statement, see [Ser02], section II.5.6, proposition 18, part (b). To deduce the second statement, note that A^{I_K} is a G_K-module, since $I_K \subset G_K$ is normal. It follows that:

$$|H^1(G_K/I_K, A^{I_K})| = |H^0(G_K, A^{I_K})|$$

By definition, $H^0(G_K, A) = A^{G_K}$. Since a fixed point for G_K is certainly a fixed point for I_K, the righthand term equals $|H^0(G_K, A)|$. □

Proposition 2.7. *Let M be a finite-dimensional $\mathbb{F}_q$-vector space with a discrete G_K-action. Then for all $i \geq 0$ we have an isomorphism:*

$$H^i(G, M) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_p \to H^i(G, M \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_p)$$

Finally, we need some basic facts about Serre fundamental characters.

Definition 2.8. Let $\pi \in \overline{K}$ satisfy π^{p^f-1}, where f is the degree of the unramified extension $K/\mathbb{Q}_p$. The character $\omega_f \colon G_K \to k^{\times}$ is defined by the condition that

$$\frac{g(\pi)}{\pi} \equiv \omega_f(g) \mod \pi,$$

for all $g \in G_K$.

Fix an embedding $\tau_0 \colon k \to \overline{\mathbb{F}}_p$, and for each i let $\tau_i = \tau_0 \circ \mathrm{Frob}$. Let $\omega_{f,i} = \tau_i \circ \omega_f = \omega_{f,0}^{p^i}$. The maps $\omega_{f,i}$ are called the *Serre fundamental characters.*

Lemma 2.9. *Let $\chi \colon G_K \to \overline{\mathbb{F}}_p$ be any character. Then there are integers $a_0, \dots, a_{f-1}$ with $1 \leq a_j \leq p$, such that*

$$\chi|_{I_K} = \omega_{f,0}^n|_{I_K},$$

where $n = \sum_{j=0}^{f-1} a_j p^j$. In other words, Serre fundamental characters generate the restriction of all characters to inertia. The integers a_j are unique if we require that $a_j \neq p$ for some j.

Proof. By continuity, the image of χ lands in some finite subfield of $\overline{\mathbb{F}}_p$. It follows that χ factors through some finite quotient. Therefore, χ is determined by its restriction to W_K^{ab}, which by class field theory is isomorphic to $K^\times$. Therefore, we only need to understand characters $\chi\colon K^\times \to \mathbb{F}_{p^n}^\times$. By proposition II.5.3 of [Neu99], we have

$$K^\times \cong (\pi) \times k^\times \times U^1,$$

where U^1 is the group of principal units. By local class field theory, the element $\pi \in K^\times$ maps to Frobenius in G_K, and Frob $\notin I_K$. It follows that $I_K \cap W_K^{\mathrm{ab}}$ corresponds under the Artin map to $k^\times \times U^1$. The group U^1 is a pro-p group, while $\mathbb{F}_{p^n}^\times$ has order $p^n - 1$, which is coprime to p. It follows that the restriction of χ to U^1 is trivial. It follows that the restriction of χ to inertia is determined by its restriction to $k^\times$. Also, the image of $k^\times$ must land in the $(p^f - 1)$-torsion subgroup of $\overline{\mathbb{F}}_p^\times$, which we can identify with $k^\times$ by taking some embedding.

Since $k^\times$ is cyclic of order $p^f - 1$, there are $p^f - 1$ homomorphisms $k^\times \to k^\times$. By proposition 3 of [Ser72], these homomorphisms exactly correspond to powers ω_f^m, for $0 \leq m \leq p^f - 1$. By composing with the embedding τ_0, we find that the restriction to inertia of the character $\chi\colon G_K \to \overline{\mathbb{F}}_p$ is of the form $\omega_{f,0}^m$ for some $m \in p^f - 1$. Note that $\omega_{f,0}$ has order $p^f - 1$, and that no two powers $\omega_{f,0}^j$ agree restricted to inertia (this is because they correspond to distinct maps $k^\times \to k^\times$, as above). It follows that m is unique if we exclude the possibility $m = p^f - 1$.

The exponent m has a unique p-adic expansion $m = \sum_{j=0}^{f-1} a_j p^j$, where $0 \leq a_j \leq p - 1$. Since the exponent is only well-defined modulo $p^f - 1$, we may instead require $1 \leq a_j \leq p$. In this representation, the cyclotomic character can be identified both with $(1, 1, \ldots, 1)$ and with $(p, p, \ldots, p)$. If we exclude the second possibility, then for reasons of cardinality the representation must be unique. $\square$

In the special case $K = \mathbb{Q}_p$, we have a character $\omega_1\colon G_{\mathbb{Q}_p} \to \mathbb{F}_p$. An important fact which we will use is that ω_1 is just the mod p cyclotomic character.

Given some χ, let $(a_0, \ldots, a_{f-1})$ denote the corresponding expansion; we call this the *tame signature* of χ. For convenience, we extend the definition of a_j to all integers by requiring its values to be periodic: if $j \equiv j' \mod p^f - 1$, then $a_j = a_{j'}$. Noting that $\mathrm{Frob}\,\chi = \chi^p$, Frob acts on the tame signature by cyclically permuting its terms:

$$\mathrm{Frob} \cdot (a_0, \ldots, a_{f-1}) = (a_{f-1}, a_0, \ldots, a_{f-2})$$

The *period* of a tame signature is defined to be the size of its orbit under the Frobenius action. For $0 \leq j \leq f - 1$, define:

$$n_j = \sum_{j=0}^{f-1} a_{i+j} p^j$$

The quantities n_j are defined so that $n_0 \equiv n_i p^i \mod p^f - 1$, and $\chi|_{I_K} = \omega_{f,i}^{n_i}|_{I_K}$.

3 Computing the jumps

Here is our main result:

Theorem 3.1 ([DDR16], Theorem 3.1)**.** *Let $d_s = \dim_{\overline{\mathbb{F}}_p} \mathrm{gr}^s(H^1(G_K, \bar{\mathbb{F}}_p(\chi)))$, for $s \in \mathbb{R}$. Then $d_s = 0$ unless $s = 0$ or $1 < s \le 1 + \frac{p}{p-1}$. Moreover, if $d_s \neq 0$ and $1 < s < 1 + \frac{p}{p+1}$, then $s = 1 + \frac{m}{p^f - 1}$ for some integer m not divisible by p. Suppose that χ has tame signature $(a_0, \ldots, a_{f-1})$, of period f', and that the integers n_i are defined as above. Then we have:*

1. *$d_0 = 1$ if χ is trivial and $d_0 = 0$ otherwise;*
2. *if $1 < s < \frac{p}{p-1}$, then*

$$d_s = \begin{cases} f/f', & \text{if } s = \dfrac{n_{i+k}}{p^f - 1} \text{ for some } i, k \text{ such that } k > 0, a_i = p, \\ & \quad a_{i+1} = \cdots = a_{i+k-1} = p - 1, \text{ and } a_{i+k} \neq p - 1, \\ 0, & \text{otherwise;} \end{cases}$$

3. *if $\frac{p}{p-1} \le s < 1 + \frac{p}{p-1}$, then*

$$d_s = \begin{cases} f/f', & \text{if } s = 1 + \frac{n_i}{p^f - 1} \text{ for some } i \text{ such that } a_i \neq p, \\ 0, & \text{otherwise;} \end{cases}$$

4. *$d_{1+\frac{p}{p-1}} = 1$ if χ is cyclotomic and $d_{1+\frac{p}{p-1}} = 0$ otherwise.*

Let d'_s denote the value given in the statement. Our general strategy is to first show that $\sum_s d'_s = d_s$, or in other words that the sum of the jumps in the filtration is the same as the claimed sum of jumps. Once this is proven, it's enough to check that for all s, $d'_s \le d_s$, because then the fact that the sums are the same implies that equality must hold. This reduces us to checking the specific values of s mentioned in the statement of the theorem.

Since Fil^s is an increasing filtration on $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$, the sum of the jumps in its dimension will just equal $\dim_{\overline{\mathbb{F}}_p} H^1(G_K, \overline{\mathbb{F}}_p(\chi))$. To determine this dimension, we use the facts about Galois cohomology quoted above. Let $q = p^f$. By proposition 2.7, we have:

$$\dim_{\overline{\mathbb{F}}_p} H^1(G_K, \overline{\mathbb{F}}_p(\chi)) = \dim_{\mathbb{F}_q} H^1(G_K, \mathbb{F}_q(\chi))$$

Therefore, we only have to compute the dimension of the cohomology group on the right. The module $\mathbb{F}_q(\chi)$, unlike $\overline{\mathbb{F}}_p(\chi)$, is finite. Therefore, we can apply proposition 2.5, and we obtain:

$$p^{-f v_p(q)} = \mathrm{EC}(\mathbb{F}_q(\chi)) = |H^0(G_K, \mathbb{F}_q(\chi))||H^1(G_K, \mathbb{F}_q(\chi)|^{-1}|H^2(G_K, \mathbb{F}_q(\chi)|$$

We have $p^{v_p(q)} = q = p^f$. It follows that $p^{-f v_p(q)} = q^{-f}$. Therefore, this result amounts to:

$$-f = \dim H^0(G_K, \mathbb{F}_q(\chi)) + \dim H^2(G_K, \mathbb{F}_q(\chi)) - \dim H^1(G_K, \mathbb{F}_q(\chi))$$

Applying proposition 2.4, we have:

$$H^2(G_K, \mathbb{F}_q(\chi)) \cong H^0(G_K, \mathbb{F}_q(\chi)')$$

By definition, $\mathbb{F}_q(\chi)' = \mathrm{Hom}(\mathbb{F}_q(\chi), \mu)$. Since $\mathbb{F}_q$ has characteristic p, the image in fact must land in μ_p, which as a Galois module is isomorphic to $\mathbb{F}_p(\mathrm{cyc})$, letting cyc denote the cyclotomic character. Thus, $\mathbb{F}_q(\chi)' = \mathrm{Hom}(\mathbb{F}_q(\chi), \mathbb{F}_p(\mathrm{cyc})) = \mathbb{F}_q(\chi^{-1} \cdot cyc)$.

Note that $H^0(G_K, \mathbb{F}_q(\chi))$ has dimension 1 if χ is trivial and dimension 0 otherwise, since the invariants form a vector subspace. The same argument shows that $H^0(G_K, \mathbb{F}_q(\chi)') = H^0(G_K, \mathbb{F}_q(\chi^{-1} \cdot \mathrm{cyc}))$ has dimension 1 if χ is cyclotomic and dimension 0 otherwise. If $p = 2$, then the cyclotomic character is trivial. Thus, the formula shows that $H^1(G_K, \mathbb{F}_q(\chi))$ has dimension f if χ is nontrivial, and $f + 2$ otherwise. If $p \neq 2$, then the cyclotomic and trivial characters are distinct. It follows that at most one of the terms H^0, H^2 has positive dimension; the dimension will be $f + 1$ if χ is trivial or cyclotomic, and f otherwise. Thus:

$$\sum_{s \in \mathbb{R}} d_S = \begin{cases} f+2, & \text{if } p = 2 \text{ and } \chi \text{ is trivial,} \\ f+1, & \text{if } p > 2 \text{ and } \chi \text{ is trivial (equivalently, cyclotomic)} \\ f, & \text{otherwise.} \end{cases}$$

A direct computation (for which we refer the reader to [DDR16]) shows that this is also the value of $\sum d'_s$. Therefore, we only need to consider the values of s in the theorem, and show for each that $d'_s \leq d_s$.

First, consider the case $s = 0$. By inflation-restriction, we have an exact sequence:

$$0 \to H^1(G_K/I_K, \overline{\mathbb{F}}_p(\chi)^{I_K}) \to H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \to H^1(I_K, \overline{\mathbb{F}}_p(\chi))$$

Recall that $\mathrm{gr}^0 = \mathrm{Fil}^0 / \mathrm{Fil}^{<0}$. We showed that $\mathrm{Fil}^{<0} = \cup_{t<0} \mathrm{Fil}^t$. Since $\mathrm{Fil}^t = 0$ for all $t < 0$, this shows that $\mathrm{Fil}^{<0} = 0$, so $\mathrm{gr}^0 \cong \mathrm{Fil}^0$. By definition, we have:

$$\mathrm{Fil}^0 = \bigcap_{u > -1} \ker \left[H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \to H^1(G_k^u, \overline{\mathbb{F}}_p(\chi)) \right]$$

Recalling that $G_K^u = I_K$ for $-1 < u \leq 0$, this means Fil^0 is the kernel of the restriction map $H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \to H^1(I_K, \overline{\mathbb{F}}_p(\chi))$. But by the inflation-restriction exact sequence above, this kernel is exactly $H^1(G_K/I_K, \overline{\mathbb{F}}_p(\chi)^{I_K})$. By proposition 2.6, this term has dimension 0 if χ is trivial and dimension 1 otherwise. Therefore, $d_0 = d'_0$.

In the remaining cases, the statement of the theorem shows that $s > 1$, and that $m = (s - 1)(p^f - 1)$ is an integer. Likewise, it shows that either $0 < m < \frac{p(p^f-1)}{p-1}$ and m is not divisible by p, or $m = \frac{p(p^f-1)}{p-1}$. For these cases, we have the following lemma:

Lemma 3.2. *Pick $\pi \in \overline{K}$ such that $\pi^{p^f-1} = -p$. There exists some unramified extension L/K of degree prime to p such that if we set $M = L(\pi)$, then $\chi|_{G_M}$ is trivial. Furthermore, $\chi = \mu\omega_{f,0}^{n_0}$ for some unramified character μ of* $\mathrm{Gal}(L/K)$.

Proof. By lemma 2.9, we know that $\chi|_{I_K} = \omega_{f,0}^{n_0}|_{I_K}$ for some n_0. It follows that the character $\mu := \chi \cdot \omega_{f,0}^{n_0}$ is unramified, i.e. trivial restricted to inertia. Since μ is continuous, we can write $\ker(\mu) = G_L$ for some finite L/K. Furthermore, since $I_K \subset \ker(\mu)$, we have $L \subset K^{\mathrm{ur}}$, meaning L is an unramified extension. By the definition of μ, we have $\chi = \mu \cdot \omega_{f,0}^{n_0}$.

Since μ is unramified, it gives a character $\overline{\mu}\colon \hat{\mathbb{Z}} \cong G_K/I_K \to \overline{\mathbb{F}}_p$. This character is continuous, so it's determined by its restriction to $\mathbb{Z} \subset \hat{\mathbb{Z}}$, which in turn is determined by the image of 1. The image of 1 in $\overline{\mathbb{F}}_p$ is some element of order a, and a is prime to p since all elements of $\overline{\mathbb{F}}_p$ have order coprime to p. This makes it clear that both $\overline{\mu}$ and μ have order a, coprime to p.

It follows that $\mathrm{Gal}(L/K)$ has order dividing a, since for any $\sigma \in G_K$, $\mu(\sigma^a) = \mu^a(\sigma) = 0$. Thus, L is an unramified extension of K of order dividing p, as desired. It remains to prove that the character is trivial restricted to G_M, where $M := L(\pi)$. The character μ is trivial restricted to G_L, by definition. The Serre fundamental characters are obtained by considering the action of G_K on the element π, so it's clear that $\omega_{f,0}^{n_0}$ is trivial restricted to $G_{K(\pi)}$. It follows that the product $\mu \cdot \omega_{f,0}^{n_0} = \chi$ is trivial on $G_L \cap G_{K(\pi)}$, which is $G_{L(\pi)} = G_M$ by Galois theory. □

Returning to the main proof, note that $\mathrm{Gal}(M/K)$ has order prime to p. It follows by Maschke's theorem that all $\overline{\mathbb{F}}_p[\mathrm{Gal}(M/K)]$-modules are semisimple, which implies that the functor $A \mapsto A^G$ is exact restricted to this category. Since Galois cohomology functors H^i are the derived functors of the functor of invariants, this shows that $H^i(\mathrm{Gal}(M/K), A) = 0$ for all $i \geq 1$, as long as A is a $\overline{\mathbb{F}}_p$-vector space. We have an inflation-restriction sequence:

$$H^1(\mathrm{Gal}(M/K), \overline{\mathbb{F}}_p(\chi)) \to H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \to H^1(G_M, \overline{\mathbb{F}}_p(\chi)) \to H^2(\mathrm{Gal}(M/K), \overline{\mathbb{F}}_p(\chi))$$

The first and last terms vanish, so we have an isomorphism:

$$H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \cong H^1(G_M, \overline{\mathbb{F}}_p(\chi))^{\mathrm{Gal}(M/K)}$$

The G_M-action is trivial, and every map $G_M \to \overline{\mathbb{F}}_p(\chi)$ factors through G_M^{ab}, since $\overline{\mathbb{F}}_p$ is an Abelian group. Thus, we have:

$$H^1(G_M, \overline{\mathbb{F}}_p(\chi))^{\mathrm{Gal}(M/K)} = \mathrm{Hom}_{\mathrm{Gal}(M/K)}(G_M, \overline{\mathbb{F}}_p(\chi))$$

This is a group of continuous homomorphisms, and as always continuous homomorphisms factor through a finite quotient. Thus, we can identify the group homomorphisms whose domain is $\mathrm{Gal}(M/K)$ with homomorphisms whose domain is W_M^{ab}. Composition with the Artin map identifies W_M^{ab} with $M^\times$; along with the fact that $\overline{\mathbb{F}}_p(\chi)$ is p-torsion, this shows:

$$\mathrm{Hom}_{\mathrm{Gal}(M/K)}(G_M, \overline{\mathbb{F}}_p(\chi)) = \mathrm{Hom}_{\mathrm{Gal}(M/K)}(M^\times/(M^\times)^p, \overline{\mathbb{F}}_p(\chi))$$

M is tamely ramified over K, so $M \subset K^{\mathrm{tame}}$, and by Galois theory this implies that $G_{K^{\mathrm{tame}}} = P_K = G^{0^+} \subset G_M$. Therefore, for all $u > 0$, $G_K^u \subset G_M$. By transitivity of the upper numbering we find that if $x \in G_M$, then $x \in G_K^u$ if and only if $x \in G_M^{u(p^f-1)}$, applying that M/K is tamely ramified of degree $p^f - 1$. Thus $G_k^u = G_M^{u(p^f-1)}$. Writing $v = \lceil u(p^f - 1) \rceil$, since $\mathrm{Gal}(M^{\mathrm{ab}}/M)$ is Abelian, it follows from the Hasse-Arf theorem that $G_M^{u(p^f-1)}$ and G_M^v have the same image in G_M^{ab}, namely $(G_M^{\mathrm{ab}})^v$. Since the Artin map is compatible with the ramification filtration, it maps $(G_M^{\mathrm{ab}})^v \cap W_M^{\mathrm{ab}}$ onto U^v. Recalling that $m = (s-1)(p^f - 1)$, it follows that an element of $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$ is trivial restricted to G_K^u for all $u > s - 1$ if and only if the corresponding homomorphism $M^\times \to \overline{\mathbb{F}}_p(\chi)$ is trivial restricted to U^{m+1}, which holds if and only if the map factors through $M^\times/(M^\times)^p U^{m+1}$ (note that for small ε, $m + 1 = \lceil (s - 1 + \varepsilon)(p^f - 1) \rceil$). The same procedure shows that a class has trivial restriction to G_K^{s-1} if and only if the corresponding homomorphism factors through $M^\times/(M^\times)^p U^m$. Using the definition of the filtration, we have:

$$\mathrm{gr}^s(H^1(G_K, \overline{\mathbb{F}}_p(\chi))) \cong \mathrm{Hom}_{\mathrm{Gal}(M/K)}(U^m/(U^m \cap (M^\times)^p)U^{m+1}, \overline{\mathbb{F}}_p(\chi))$$

Consider cases 2 and 3 of the theorem, where $m < \frac{p(p^f-1)}{p-1}$ and m is not divisible by p. We claim that $U^m \cap (M^\times)^p \subset U^{m+1}$. If not, there exists some $x^p \in U^m \cap (M^\times)^p$ such that $x^p \notin U^{m+1}$. This implies that $v_\pi(x^p - 1) = m$.

Let $t = v_\pi(x-1)$. Since $v(x^p) = 0$, we know that $v(x) = 0$ as well, so we can write $x = yz$, where y is a Teichmüller root of unity and $z \in U^1$. We have $x^p = y^p z^p \in U^m$, so we know $y^p = 1$. Since the group of Teichmüller roots has order prime to p, this implies $y = 1$, so $x \in U^1$. Thus, $t > 0$. Applying this, write $x = 1 + y\pi^t$, where $y \in O_M^\times$. We have:

$$x^p - 1 = (1 + y\pi^t)^p - 1 = \sum_{i=1}^{p} \binom{p}{i} y^i \pi^{ti} = py\pi^t + \sum_{i=2}^{p} \binom{p}{i} y^i \pi^{ti}$$

We have $v_\pi(py\pi^t) = t + p^f - 1$. For $2 \leq i < p$, we have:

$$v_\pi\left(\binom{p}{i} y^i \pi^{ti}\right) = p^f - 1 + ti \geq t + p^f - 1$$

Finally, for $i = p$, we have $v_\pi(y^p \pi^{tp}) = tp$. It follows that $m = v_\pi(x^p - 1) \geq \min(t + p^f - 1, tp)$, and that equality holds unless $t + p^f - 1 = tp$. First, suppose that $t + p^f - 1 > tp$. Then $m = tp$, but this contradicts the fact that in the case under consideration, m is not divisible by p. Otherwise, we have $t + p^f - 1 \leq tp$. This rearranges to $t(p-1) \geq p^f - 1$, $t \geq \frac{p^f - 1}{p-1}$. Thus:

$$m \geq \min(tp, t + p^f - 1) = t + p^f - 1 \geq \frac{p^f - 1}{p-1} + p^f - 1 = \frac{p(p^f - 1)}{p-1}$$

But this contradicts the assumption we made (in cases 2 and 3) that $m < \frac{p(p^f-1)}{p-1}$. We conclude by contradiction that indeed $(M^\times)^p \cap U^m \subset U^{m+1}$. It follows that:

$$\mathrm{gr}^s(H^1(G_K, \overline{\mathbb{F}}_p(\chi)) \cong \mathrm{Hom}_{\mathrm{Gal}(M/K)}(U^m/U^{m+1}, \overline{\mathbb{F}}_p(\chi))$$

Let l be the residue field of L, and define a map $l \to U^m/U^{m+1}$ by $x \mapsto 1 + x\pi^m$. This is an isomorphism of groups, and we claim it gives an isomorphism between the $\mathrm{Gal}(M/K)$-modules $l \otimes_k k(\omega_f^m) =: l(\omega_f^m)$ and U^m/U^{m+1}. Indeed, for any $g \in \mathrm{Gal}(M/K)$, we have:

$$g(1 + x\pi^m) = 1 + g(x)g(\pi)^m$$

Meanwhile, g acts on $l(\omega_f^m)$ by $x \mapsto g(x)\omega_f^m(g)$. Since $\omega_f^m(g)\pi^m = g(\pi)^m$, this shows that the claimed isomorphism holds.

Write $\tau := \omega_f^m$. By the normal basis theorem, we find that $l \cong k[\mathrm{Gal}(l/k)]$, as $\mathrm{Gal}(l/k)$-modules. Since $\mathrm{Gal}(M/K)$ acts via the quotient mapping onto $\mathrm{Gal}(l/k)$, the same isomorphism is also an isomorphism of $\mathrm{Gal}(M/K)$-modules.

The space $k[\mathrm{Gal}(l/k)] \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ is isomorphic to:

$$\bigoplus_{\sigma \in S} \langle x_\sigma \rangle$$

Here, σ ranges over embeddings $l \to \overline{\mathbb{F}}_p$, and the Galois action comes from the action of $\mathrm{Gal}(l/k)$ on these embeddings. We can split this into distinct $\mathrm{Gal}(M/K)$-stable subspaces, in the following way:

$$\bigoplus_{\sigma \in S} \langle x_\sigma \rangle = \bigoplus_{i=0}^{f-1} \left(\bigoplus_{\sigma \in S_i} \langle x_\sigma \rangle \right)$$

Here, S_i is the set of embeddings $l \to \overline{\mathbb{F}}_p$ which extend τ_i (recall that the maps τ_i are the distinct embeddings $k \to \overline{\mathbb{F}}_p$). The structure of this direct sum as a k-vector space is characterized by $\lambda(v) = \tau_i(\lambda) \cdot v$, for all $\lambda \in k$, and $v \in \oplus_{\sigma \in S_i} \langle x_\sigma \rangle$. It follows that when we twist by the character $k(\tau) = k(\omega_f^m)$, we obtain:

$$l(\omega_f^m) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \cong \bigoplus_{i=0}^{f-1} \left(\bigoplus_{\sigma \in S_i} \overline{\mathbb{F}}_p(\omega_{f,i}^m) \right)$$

Here, the action is defined by $g(x_\sigma)_\sigma = \omega_{f,i}^m(g)(x_{\sigma \circ g})_\sigma$. Note that $\oplus_{\sigma \in S_i} \overline{\mathbb{F}}_p$ is just the induction from $\mathrm{Gal}(M/L)$ to $\mathrm{Gal}(M/K)$ of the trivial $\mathrm{Gal}(M/L)$-representation $\overline{\mathbb{F}}_p$, and is therefore isomorphic to $\overline{\mathbb{F}}_p[\mathrm{Gal}(L/K)]$. Since $\mathrm{Gal}(L/K)$ has order prime to p and is Abelian, its group algebra over $\overline{\mathbb{F}}_p$ splits as a direct sum of characters, and we can write:

$$\bigoplus_{\sigma \in S_i} \overline{\mathbb{F}}_p = \bigoplus_{\mu} \overline{\mathbb{F}}_p(\mu), \bigoplus_{\sigma \in S_i} \overline{\mathbb{F}}_p(\omega_{f,i}^m) = \bigoplus_{\mu} \overline{\mathbb{F}}_p(\mu \omega_{f,i}^m)$$

Here, μ ranges over characters $\mathrm{Gal}(L/K) \to \overline{\mathbb{F}}_p^\times$. Combining this with our earlier isomorphism $U^m/U^{m+1} \cong l(\omega_f^m)$, we have:

$$(U^m/U^{m+1}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \cong \bigoplus_{i=0}^{f-1} \bigoplus_{\mu} \overline{\mathbb{F}}_p(\mu \omega_{f,i}^m)$$

Recall that in this case, $d_s = \dim(\mathrm{Hom}_{\mathrm{Gal}(M/K)}(U^m/U^{m+1}, \overline{\mathbb{F}}_p(\chi)))$, which will just equal the multiplicity of $\overline{\mathbb{F}}_p(\chi)$ in the representation $U^m/U^{m+1} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. If $m \equiv n_i \mod (p^f - 1)$, then $\omega_{f,i}^m = \omega_{f,0}^{n_0}$, and there exists some $\mu \colon \mathrm{Gal}(L/K) \to \overline{\mathbb{F}}_p^\times$ with $\mu \omega_{f,i}^m = \chi$. This μ is clearly unique. On the other hand, if $m \not\equiv n_i \mod (p^f - 1)$, then there is no μ with $\mu \omega_{f,i}^m = \chi$. It follows that d_s is the number of i such that $m \equiv n_i \mod (p^f - 1)$. Working directly from the definition, we see that in cases 2 and 3, $d'_s \leq d_s$, recalling that f' is the period of the tame signature.

Finally consider case 4, where $s = 1 + \frac{p}{p-1}$ and $m = \frac{p(p^f-1)}{p-1}$. Again we only need to prove that $d'_s \leq d_s$. Therefore, we can assume χ is cyclotomic, because otherwise $d'_s = 0$ and the result is trivial. When χ is cyclotomic $d'_s = 1$, so we only need to show $d_s \geq 1$.

Suppose $x \in U^{m+1}$. Then $v(\log x) \geq m + 1$, so $v(p^{-1} \log x) \geq m + 1 - (p^f - 1) = 1 + \frac{p^f-1}{p-1}$. Since the ramification degree of $M/\mathbb{Q}_p$ is $p^f - 1$, it follows from proposition II.5.5 of [Neu99] that exp and log define mutually inverse isomorphisms between $(p^{-1} \log x) O_M$ and $1 + (p^{-1} \log x) O_M$. Thus, we have $(\exp(p^{-1} \log x))^p = \exp(\log x) = x$. This shows explicitly that $x \in (M^\times)^p$, so $U^{m+1} \subset (M^\times)^p$. It follows that every homomorphism $(M^\times)/(M^\times)^p \to \overline{\mathbb{F}}_p(\chi)$ factors through $M^\times/(M^\times)^P U^{m+1}$, since $(M^\times)^p U^{m+1} = U^{m+1}$. Therefore, it follows from the characterization deduced above that $\mathrm{Fil}^s(H^1(G_K, \overline{\mathbb{F}}_p(\chi)) = H^1(G_K, \overline{\mathbb{F}}_p(\chi))$. In order to show that $d_s \geq 1$, we only need to show that $\mathrm{Fil}^{<s} H^1(G_K, \overline{\mathbb{F}}_p(\chi))$ is not all of $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$. By definition, $\mathrm{Fil}^{<s}$ is the kernel of restriction to $G_K^{s-1} = G_K^{p/p-1}$, so we only need to find a cohomology class whose restriction to $G_K^{p/p-1}$ is nontrivial. Since χ is the cyclotomic character, we can think of it as a character of $G_{\mathbb{Q}_p}$. Furthermore, $G_K^{p/(p-1)} = G_{\mathbb{Q}_p}^{p/(p-1)}$, applying transitivity of upper numbering along with the fact that K is unramified. The restriction maps give us a commutative diagram:

$$
\begin{array}{ccc}
H^1(G_{\mathbb{Q}_p}, \overline{\mathbb{F}}_p(\chi)) & \longrightarrow & H^1(G_{\mathbb{Q}_p}^{p/(p-1)}, \overline{\mathbb{F}}_p(\chi)) \\
\Big\downarrow & & \Big\| \\
H^1(G_K, \overline{\mathbb{F}}_p(\chi)) & \longrightarrow & H^1(G_K^{p/(p-1)}, \overline{\mathbb{F}}_p(\chi))
\end{array}
\tag{3.0.1}
$$

If the bottom map is trivial, then the top map must be trivial as well. Therefore, we may assume that $K = \mathbb{Q}_p$. Recall that we constructed an unramified character μ of $\mathrm{Gal}(L/K)$ such that $\chi = \mu\omega_{f,0}^{n_0}$; our construction showed that $[L : K] = |\mu|$, where $|\mu|$ is the order of μ. In this case, $\chi = \omega_1$, so $\mu = 1$, and it follows that we may take $L = \mathbb{Q}_p$, $M = \mathbb{Q}_p(\pi)$.

Recall that ω_1 is the mod p cyclotomic character. It follows that its kernel is exactly $G_{\mathbb{Q}_p(\zeta_p)}$. However, by definition, its kernel is also $G_{\mathbb{Q}_p(\pi)}$, so by Galois theory we have $M = \mathbb{Q}_p(\pi) = \mathbb{Q}_p(\zeta_p)$.

Suppose that $x \in U^1$. We claim $x^p \in U^{p+1}$. Writing $x = 1 + \pi y$ for some $y \in O_M$, we have:

$$
x^p = (1 + \pi y)^p = 1 + p\pi y + \sum_{i=2}^{p-1} \binom{p}{i} \pi^i y^i + \pi^p y^p
$$

We have $v(p\pi y) = v(\pi^p y^p) = p$, while all other terms in the sum have higher valuation. Therefore, in order to show that $x^p \in U^{p+1}$, we only need to show that $\pi^p y^p \equiv p\pi y \mod \pi^{p+1}$. If $z \equiv y \mod \pi$, then $p\pi y \equiv p\pi z \mod \pi^{p+1}$, and likewise $\pi^p y^p \equiv \pi^p z^p \mod \pi^{p+1}$. Therefore, we may assume that y is a Teichmüller root of unity. But then $y^p = y$. We have $\pi^p = -p\pi$, so $\pi^p y^p = -p\pi y$, and the result follows. By the isomorphism we established earlier in the proof, we have:

$$
\mathrm{gr}^s(H^1(G_{\mathbb{Q}_p}, \overline{\mathbb{F}}_p(\chi)) \cong \mathrm{Hom}_{\mathrm{Gal}(M/\mathbb{Q}_p)}(U^p/U^{p+1}, \overline{\mathbb{F}}_p(\chi))
$$

In this case of the proof we have $l = \mathbb{F}_p$, so our result above shows that $U^p/U^{p+1} \cong \mathbb{F}_p(\chi)$. But the embedding $\mathbb{F}_p(\chi) \hookrightarrow \overline{\mathbb{F}}_p(\chi)$ is clearly $\mathrm{Gal}(M/\mathbb{Q}_p)$-equivariant, so this space is nonzero. It follows that $d_s \geq 1$, completing the proof. ☐

Finally, note that this does answer the question we started with. For $u > 0$, there exists some representation ρ with a jump at u if and only if the filtration on $H^1(G_K, \overline{\mathbb{F}}_p(\chi))$ has a jump at u. These are exactly the cases outlined in cases 2, 3, and 4 of the theorem.

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