

Newton's Numerator in 1685

A Year of Gestation

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**Toward corroboration of a Howard Stein
conjecture at the end of his '*From the
Phenomena of Motions to the Forces of
Nature*': *Hypothesis or Deduction?* (1990/91)**

The Numerator in the 1687 *Principia*

Prop 5: The circumjovial planets gravitate toward Jupiter, the circumsaturnian planets gravitate toward Saturn, and the circumsolar planets gravitate toward the Sun, and by the force of their gravity they are always drawn back from rectilinear motions and kept in curvilinear orbits.

Corol 1: Therefore there is gravity toward all planets universally. For no one doubts that Venus, Mercury, and the rest are bodies of the same kind as Jupiter and Saturn. **And since (by the third law of motion) every attraction is mutual**, Jupiter will gravitate toward its satellites, Saturn toward its satellites, the Earth will gravitate toward the Moon, and the Sun toward all the primary planets.

Prop. 6: All bodies gravitate toward each of the planets, and at any given distance from the center of any one planets the weight of any body whatever toward that planet is proportional to the quantity of matter which the body contains.

Corol.4 (5 in subseq. ed.): The force of gravity is of a different kind from the magnetic force. For magnetic attraction is not proportional to the matter attracted. Some bodies are attracted more, and others less, while most bodies are not attracted....

Prop. 7: Gravity exists in all bodies universally and is proportional to the quantity of matter in each.

We have already proved that all planets are heavy toward one another and also that the gravity toward any one planet, taken by itself, is inversely as the square of the distance of places from the center of the planet. And it follows (by Book 1, Prop. 69) that **the gravity toward all the planets is proportional to the matter in them.**

Corol. 1: Therefore the gravity toward the whole planet arises from and is compounded of the gravity toward the individual parts. We have examples of this in magnetic and electric attractions. For every attraction toward a whole arises from the attractions toward the individual parts....

COTES'S OBJECTION TO PROP. 5, COROL. 1

But in the first corollary of the 5th Proposition I meet with a difficulty, it lies in these words [*Et cum attractio omnis mutua sit*]. I am persuaded they are then true when the Attraction may properly be so called, otherwise they may be false. You will understand my meaning by an Example. Suppose two Globes *A* & *B* placed at a distance from each other upon a Table & that whilst the Globe *A* remains at rest the Globe *B* is moved towards it by an invisible Hand; a by-stander who observes this motion but not the cause of it, will say that the globe *B* does certainly tend toward the center of Globe *A*, & thereupon he may call the force of the invisible hand the centripetal force of *B* & the attraction of *A* since the effect appears the same as if it did truly proceed from a proper and real Attraction of *A*.

Cotes to Newton, 18 Mar. 1713

... Now the mutual & mutually equal attractions of bodies is a branch of the third Law of Motion & how this branch is deduced from Phenomena you may see in the end of the Corollaries of the Laws of Motion, pag. 22. If a body attracts another body **contiguous** to it & is not mutually attracted by the other: the attracted body will drive the other before it & both will go away together with an accelerated motion in infinitum, as it were by a self moving principle, contrary to the first law of motion, whereas there is no such phenomenon in all nature.

Newton to Cotes, 28 Mar. 1713

EULER ON THE MASS OF THE ATTRACTING BODY

As for the cause of gravity which I have rejected, I consider the objection that the attraction must not necessarily be proportional to the masses, to be of no great importance, as it is **still not decided by any single phenomenon that the attractive forces of heavenly bodies are proportional to their masses.** On the contrary, Newton tried to determine the masses on this basis since there is no other way of specifying them. As soon as one now places the statement that the attractive forces are proportional to the masses (**which is founded on a crude hypothesis**) in doubt, this objection against my idea is eliminated.

Euler to Mayer
25 December 1751

From Stein (1990/91)

But the third law of motion does not tell us that whenever one body is urged by a force directed towards a second, that second body experiences an equal force towards the first – it tells us, rather, that whenever one body is acted upon *by* a second, the second body is subject to a force equal in magnitude and opposite in direction.... Furthermore, it must not be thought that ... the only plausible subject of the reaction to gravitational force towards a planet is the planet itself. On the contrary, the very widespread view of Newton's time that one body can act upon another only by contact ... makes for precisely the opposite assessment: that it is far-fetched to apply the third law in the fashion Newton does.

The engine that drives the enormous step to the law of universal gravitation ... is a certain conception of the character of a force of nature or natural power, such as gravitation: namely, that a force of nature is *a force of interaction*; and that such a force is characterized by a *law of interaction*: a law in which the interacting bodies enter altogether symmetrically.

For clearly, in so far as the “deduction” validates what I have called Newton's speculative application of the third law of motion, it also contributes evidence for the cogency of the general conception of the natural powers that lies behind the application: that is, as I would put it, it “proves,” besides the metaphysics of space and time, the general metaphysics of nature expressed in the introductory sections of the *Principia* (and in the preface to the first edition). **I believe that this whole conception of the constitutional frame of nature was actually developed by Newton *at the same time* that he was discovering the law of gravity.**

The Sequence of Sources

De motu corporum in gyrum – in Newton’s hand, copy sent to Halley in November 1684

De motu sphaericorum Corporum in fluidis – preceding augmented, to include relativity and center of gravity principles, plus “Copernican scholium” (first in print by Rouse Ball in 1893)

Correspondence with Flamsteed, late Dec. 1684 to 27 January 1685 – focused on comet of 1680-81 and precision of $3/2$ power proportions

De motu corporum in mediis regulariter cedentibus – heavily reworked, consisting of 18 definitions, 5 laws (including Law 3), 2 lemmas

Insertion on verso side of first folio of preceding – includes Definition of *quantity or amount of matter* with account of 2 pendulum experiment

De motu Corporum, Liber Primus – surviving fragment consists of 24 Props. starting from Prop. 1, plus initial drafts of what were later numbered Props. 64 and 66 (with but 1 corollary) through 75

De motu Corporum, Liber Secundus – complete manuscript, laid out in the manner of Descartes’ *Principia*, consisting of 81 much reworked Articles, ending with a method for comets that proved inadequate

Correspondence with Flamsteed, 19 September to 10 October 1685 – focused on comet of 1680-81

De motu Corporum, Definitiones – single folio page in Newton’s hand, where ‘*massa*’ and ‘*inertia*’ first appear

De motu Corporum, Liber Primus – expanded version, includes drafts of Definitions, Scholium; 3 Laws, 6 Corollaries, Scholium (with pendulum impact experiment); and what became Sections 1 through 10

Manuscript for Book 1, sent to Halley in April 1686 – as published save for minor revisions and insertion in Prop. 91 of 2 corol. on spheroids

N.B. Newton was away from Cambridge from 27 March to 11 April and from 11 to 20 June 1685.

From ... *in mediis regulariter cedentibus*

Def. 11. The quantity of motion is that which arises from the velocity and the quantity of a body in translation [*corporis translati*] jointly. Moreover, the quantity of a body is to be reckoned [*aestimatur*] from the amount [*copia*] of the corporeal matter, which is usually proportional to its gravity [*gravitati*]. The oscillations of two equal pendulums with bodies of equal heaviness [*ponderis*] are counted, and the amount [*copia*] of matter in each will be reciprocally as the number of oscillations made in the same time.

From a draft on the verso side of the first folio of
... *in mediis regulariter cedentibus*, in Newton's hand

7. By the heaviness [*pondus*] of a body I understand the quantity [^]or amount [*copiam*][^] of matter [^]moved[^] apart [*abstracta*] from considerations of gravitation [*gravitationis*] as often as it is not a matter of gravitating bodies [*de gravitantibus non agitur*]. To be sure, the heaviness [*pondus*] of a gravitating body [*gravitantium*] is proportional to its quantity of matter, [^]and the agreement [*analogia*] legitimates [*licet*] setting forth [*exponere*] and designating each by the other. The agreement in fact [*vero*] is to be gathered [*colligitur*] as follows.[^] The oscillations of two equal pendulums of the same heaviness [*ponderis*] are counted and the amount [*copia*] of matter in each will be reciprocally as the number of oscillations made in the same time. Moreover, experiments carefully [*diligenter*] made on gold, silver, lead, glass, sand, [^]common salt, [^]water, wood, and wheat always led to the same number of oscillations. On account of this agreement [^]and lacking a more convenient word[^] I set forth and designate quantity of matter by heaviness [*pondus*] even when gravitation [*gravitatio*] is not being considered.

From *De Motu Corporum, Liber Secundus* [Add. 3990]

18. *Another agreement [analogia] between forces and bodies. It is proved for heavenly bodies [coelestibus]*

A second agreement between the forces and the attracted bodies is akin to the one just described. Since the action of the centripetal force upon the planets decreases in the duplicate ratio of the distance, and the periodic time is increased in the sesquialteral ratio, it is manifest that if equal planets were equally distant from the Sun, their actions would be equal and their periodic times would be equal, and that **if unequal planets were at equal distances, their [^]collective [collectitiae], [^]actions would be as the [pondera] [^]bodies, [^]of the planets. For actions that were not as the [pondera] [^]bodies to be moved, [^]could not draw those [pondera] [^]bodies, [^]equally back from the tangents of the Orbits and cause revolutions to be completed in equal times in Orbits that are also equal. But neither could the motions of the satellites of Jupiter be so regular if the circumsolar force were not exerted equally upon Jupiter and all the satellites in proportion to their weights [ponderum]. And the same is true of Saturn and its satellite and also of the earth and our Moon, as (from prop. 35, corol. 2 and 3) is manifest ~~and soon will be made more fully clear.~~ At equal distances, therefore, there is an equal action of centripetal force upon all the planets in proportion to their [ponderum] [^]bodies or quantities of matter in the bodies, [^]and thus also upon all the particles of that [ponderis] [^]quantity, [^]of which the planets are composed. For if the action were greater upon particles of one kind of matter, and less upon those of another, than in proportion to the [ponderum] [^]quantity of matter, [^]the action upon the planets would also be greater or less not only in proportion to the [ponderum] [^][corporum] quantity, [^]but also in accordance with the kind of matter, which would be found more abundantly [copiosius] in one body and more sparingly in another.**

Newton's Goal for Natural Philosophy

Sic etiamsi colores ad Physicam pertineant, eorum tamen scientia pro Mathematica habenda est, quatenus ratione mathematica tractantur. Imo vero cum horum accurata scientia videatur ex difficillimis esse quae Philosophus desideret; **spero me quasi exemplo monstraturum quantum Mathesis in Philosophia naturali valeat**; et exinde ut homines Geometras ad examen Naturae strictius aggrediendum, & avidos scientiae naturalis ad Geometriam prius addiscendum horter; ut ne priores suum omnino tempus in speculationibus humanae vitae nequaquam profuturis absumant, neque posteriores operam praepostera methodo usque navantes, a spe sua perpetuo decendant: Verum ut Geometris philosophantibus & Philosophis exercentibus Geometriam, **pro conjecturis et probabilibus quae venditantur ubique, scientiam Naturae summis tandem evidentiis firmatam nanciscamur.**

Optical Lectures, Lect. 3, 1670-72

... Thus although colors may belong to physics, the science of them must nevertheless be considered mathematical, insofar as they are treated by mathematical reasoning. Indeed, since an accurate science of them seems to be one of the most difficult that philosophy is in need of, **I hope to show – as it were, by my example – how valuable mathematics is in natural philosophy.** I therefore urge geometers to investigate nature more rigorously, and those devoted to natural science to learn geometry first. Hence the former shall not entirely spend their time in speculations of no value to human life, nor shall the latter, while working assiduously with a preposterous method, perpetually fall short of their goal. But truly with the help of philosophical geometers and geometrical philosophers, **instead of the conjectures and probabilities that are being blazoned about everywhere, we shall finally achieve a science of nature supported by the greatest evidence.**

From a letter of 27 December 1684 from Flamsteed to Newton

“In the *Transactions* of November last I published the Eclipses of the Satellites for all the next year anywhere visible.... I use their motions altogether equable only allowing Roemer’s equation of light, without which allowance the error of my tables would be above 10 minutes of time. **Now it seems strange the moon’s motion should be so perplexed with inequalities and these, for ought I can perceive yet, except in the 2nd, wholly free from them** and I have some reason to think the errors I meet with in my numbers for the 2nd Satellite may partly proceed from my having allowed its orbit to lie in the same plane with the orbits of the other 3...” [spelling modernized]

N.B. Sometime in the first quarter of 1685 – at least before he had reached Article 18 of *Liber Secundus* – Newton added a Proposition 35 (which became Prop. 65 in the *Principia*) and 21 Corollaries to the immediately following Proposition in *Liber Primus*. The original text of this insert no longer exists, only the text as it appears in the first edition of the *Principia*. **Corollaries 14 through 17 resolve Flamsteed’s quandary, taking advantage of the quantification of the accelerative tendencies toward Jupiter and the Sun provided by the magnitudes of $[a^3/P^2]$ for each respectively.**

Newton, in Early 1685 [*Liber Primus*]

Prop. 3. Every body which by a radius drawn to another body as center, **moving in any way whatsoever**, describes areas proportional to the times is urged by a force compounded of the centripetal force tending to the other body and of the force by which the second body is, **in proportion to its own bulk** [*pro mole suo*], urged.

For (by Corollary 6 of the Laws) if a new force which is equal and contrary to that by which the second body is urged shall press each body, the first body will proceed to describe around the second body the same areas as before; however, the force by which the second body was urged will now be annulled by the force equal and contrary to it, and in consequence (by Law 1) that second body will either stay at rest or move uniformly on in a straight line, while the first body, urged by the difference of the forces, will proceed to describe areas proportional to the times around the second body. Therefore (by Theorem 2) the difference of the forces tends to that second body as the center. Q.E.D.

Corol. 1. Hence, if one body by a radius drawn to a second one describes areas proportional to the times, this body will be urged by no force except that compounded of the centripetal force tending toward the other body and of every force which acts on the second body and is understood to act equally on each (**in proportion to their bodily bulk** [*pro mole corporum*]) and following parallel lines. For the addition and subtraction of forces occurs in this Theorem along linear alignments, as is set out in Corol. 1 to the Laws.

Corol. 2. And with the same suppositions, if the areas be proportional *quamproxime* to the times, that common force either acts equally on each body *quamproxime*, or acts along [*secundum*] lines *quamproxime* parallel, or is exceedingly slight if it be compared with the centripetal force tending to the second body.

19. *It is proved for terrestrial bodies [terrestribus]*

I have actually tested this proportion with the greatest exactness as possible in different kinds of bodies that exist on our Earth. The action of a circumterrestrial force that is proportional to the bodies to be moved will move them in equal times with equal velocity (by law 2) and will make all bodies that are let fall descend through equal spaces in equal times and will also make all bodies suspended by equal cords oscillate in equal times. If the action is greater, the times will be smaller, and if the action is smaller, the times will be greater. Others have long since observed that all bodies descend in equal times (at least if the very small resistance of air is removed), and it is possible to discern the equality of the times to the highest degree of accuracy in pendulums. I have tested this with gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I got two equal wooden boxes. I filled one with wood and I suspended the same weight [*pondus*] of gold (as exactly [*exacté*] as I could) at the center of oscillation of the other. The boxes, hanging by equal eleven-foot cords, made pendulums entirely like one another with respect to their weight [*pondus*], shape, and air-resistance. Then when placed close to each other, they kept swinging back and forth together with equal oscillations for a very long time. Accordingly, the amount [*copia*] of matter in the gold [^](by Prop. ____)[^] was to the amount of matter in the wood as the action of the motive force [*vis motricis*] upon all the gold to this action upon all the wood – that is, as the weight [*pondus*] of one to the weight [*pondus*] of the other. And so for all the others. In these experiments, in bodies of the same weight [*pondus*], a difference of matter that would be even less than a thousandth of the whole could have been clearly noticed. ~~Because of this agreement, I have throughout designated the quantity of matter in each individual body by the word *pondus*, using the name of the measure for the thing measured, as is the common custom.~~

A Strange Contrast

On the one hand:

The (inverse-square) centripetal accelerative tendencies [*conatus*] toward the Earth, Sun, Jupiter, and Saturn at equal distances from their centers do not vary from one body to another.

On the other hand:

The force required to offset the accelerative tendency toward the Earth – and presumably too toward the Sun, Jupiter, and Saturn – varies from one body to another, namely as their weight.

The resistance to changes of speeds of bodies colliding with one another depends on their bulk [*moles*] – the greater the bulk, the less the change of speed.

The deceleration experienced by bodies moving through the air depends not only on their velocity and shape, but also on their weight – the greater the weight, the less the deceleration.

The tension in a string retaining a body in uniform circular motion varies not only as the geometry of the motion – the square of the arc length in a given time divided by the radius – but also as the weight of the body.

“Hypoth. 2. Every body by its inherent force [*vis insita*] alone proceeds uniformly *in infinitum* in a straight line unless something from without impedes it [*aliquid extrinsecus impediatur*].”

How do the forces of gravity, unlike forces of any other kind, adjust themselves – in Émilie Du Châtelet’s phrasing, “*se proportionne*” – to the quantities of matter in the bodies on which they act?

20. *The unanimity of the agreements [Analogiarum consensus]*

And since the action of centripetal force upon the attracted body [*corpus attractum*], at equal distances, is proportional to the matter in this body, **it is reasonable also to grant** [*rationi etiam consentaneum est*] that it is proportional as well to the matter in the attracting body [*corpore trahente*]. For the action is mutual, and causes the bodies by a mutual endeavor [*conatu mutuo*] (by Law 3) to approach each other, and accordingly the action in one body must necessarily be in conformity with the action in the other. One body can be considered as attracting [*atrahens*], and the other as attracted [*attractum*], but this distinction is more mathematical than natural. The attraction is really that of either of the two bodies towards the other, [^]and thus is of the same kind in each of the bodies[^].

21. *And their coincidence [Et coincidentia]*

And hence it is that the attractive force is found in both bodies. The Sun attracts [*trahit*] Jupiter and the other Planets, Jupiter attracts its Satellites and similarly the Satellites act on one another and on Jupiter, and all the Planets act on one another. **And although, in a pair of Planets, the action of each on the other can be distinguished and can be considered as paired actions by which each attracts [*trahi*] the other, yet inasmuch as these are actions between two bodies, they are not two but a simple operation between two termini.** Two bodies can be drawn [*trahi*] to each other by the contraction of a single rope between them. The cause of the action is two-fold, namely the disposition of each of the two bodies; the action is likewise two-fold, insofar as it is upon two bodies; but insofar as it is between two bodies it is a simple and single action. **There is not, for example, one operation by which the Sun attracts [*trahit*] Jupiter and another operation by which Jupiter attracts the Sun, but a single operation by which the Sun and Jupiter endeavor to approach each other.** By the action by which the Sun attracts Jupiter, Jupiter and the Sun endeavor to approach each other (by Law 3), and by the action by which Jupiter attracts the Sun, Jupiter and the Sun also endeavor to approach each other. Moreover, the Sun is not attracted [*attrahitur*] by a twofold action towards Jupiter, nor is Jupiter attracted by a twofold action towards the Sun, but there is one action between them by which both approach each other. Iron attracts [*trahit*] a Loadstone [*magnetum*] just as much as a Loadstone attracts iron. For any iron in the vicinity of a Loadstone attracts other iron also. But the action between the Loadstone and the iron is simple, and natural philosophers consider it as simple; the operation of the iron upon the Loadstone is the very operation of the Loadstone between itself and the iron, by which both endeavor to approach each other. This is manifest from the fact that if the loadstone is removed, nearly the whole force of the iron ceases. **In this way conceive that a simple operation, arising from the concurring [*conspirante*] nature of two Planets, is exerted between them; then this operation will be the same with respect to both and thus, being proportional to the matter in one of them, will be proportional to the matter in the other.**

Deleted in its entirety: {23. *Forces tend toward all terrestrial bodies*

Let ABCD designate the globe of the earth cut by two planes that are parallel and equally distant from the center on both sides. Since the middle part AHEDIB is pressed equally on both sides by the weights [*ponderibus*] of the outermost parts AHEF and BIDC and since, because of the equality of the pressures, the middle part remains in equilibrium, it is manifest that if either part were somewhat raised by some force applied from outside and were slowly withdrawn, the middle part HI would yield to the urging weight of the other part BIDC and would tend toward the withdrawn part. And accordingly, if the part FHG and the sum of the parts HI and IC were forcefully [*violenter*] held back at some certain distance from each other and then were let go simultaneously, both bulks [*moles*] and the part FH and the bulk [*moles*] HC would rush toward each other and thus they have the power of mutual gravitation. The bulk [*moles*] HC gravitates towards the part FH equally as much as the part FH gravitates towards the bulk [*moles*] HC, because both fall towards each other.

{Inserted by Newton on the facing verso sheet: {It is not legitimate [*non licet*] to imagine that the gravitation of each of the two takes place toward ~~some third~~ another place distinct from the two bodies. For the imagined space is similar and does not have any [^]specific[^] point towards which gravitation takes place more than any other. If the whole earth were moved out of its place, there is no doubt that its parts would even then gravitate towards its center [^]and would not seek the middle (which is now put out of the center) of its former place[^]. For the properties [*affectiones*] and operations of bodies depend on the bodies and thus will not remain in spaces out of which bodies are moved, but will accompany bodies when they are transferred. Magnetic force follows a magnet, electric force follows amber, and centripetal force a Planet. And similarly the forces with which the bodies FH and HC fall towards each other ~~follow~~ accompany these bodies when they have been drawn apart. ~~Further~~ As long as the middle part HI is balanced by the contrary sections of the outermost parts FH and IC, it yields to the weight of the part IC, the other part FH having been withdrawn. But if that part FH were moved further away and its action on the bulk HC ceased because the distance was too great, it should not be believed that the part HI would now yield more to the weight of IC and would flee indefinitely with a motion always accelerated. Perpetually accelerated motion in a straight line must by no means be granted to the forces of nature. That [^]by the first law[^] denied, the body HC will be at rest, and thus by falling towards the body FH when it is near demonstrates its gravitation attraction [*attractionem*]-~~and by not falling towards FH when it is distant, demonstrates that its attractive force is diminished as a result of increased distance.~~ }

... The parts therefore urge each other equally by their weights, that is, are attracted towards each other equally (as the third Law requires) and thus if drawn apart from each other and let go would fall towards each other with velocities that would be reciprocally as the bodies.... It is legitimate [*licet*] to test and observe all this in a magnet.... }end deletion

23. *That forces proportional to the quantity of matter nevertheless tend towards all terrestrial bodies*

Let ABCD designate the globe of the earth, cut by any plane AC into two parts ACB and ACD. The part ACB, by leaning upon the part ACD, presses it with all its weight [*pondere*]. And the part ACD cannot sustain this pressure and remain unmoved except by an equal endeavor in the opposite direction. The parts, therefore, urge each other equally by their weights [*ponderibus*], that is, are drawn [*trahentur*] equally toward each other (as Law 3 requires), and hence, if drawn apart [*distractae*] from each other and released, they would each fall to the other with velocities that would be reciprocally as the bodies. It is legitimate [*licet*] to test and to observe all this in a loadstone [*magnete*]. Let ACB now designate some small body on the surface of the earth; and since *velocitas* the mutual attractions of this particle and the rest of the earth ACD are equal, and since the attraction of the particle toward the earth (that is, its weight [*pondus*]) is as the matter of the particle (as proved by the pendulum experiment), the attraction of the earth toward the particle will also be as the matter of the particle, and thus the attractive force of every terrestrial body is as the quantity of matter in it individually [*in singulis*].

24. *It is proved that these same forces tend toward celestial bodies*

Forces, moreover, that are as the matter in terrestrial bodies of all forms, and thus are not changed with the forms, must be found in all bodies universally [*universis*], celestial as well as terrestrial, and must in all of them be proportional to the matter because all these differ not in the kind of substance, but only in forms and modifications. This is proved also for celestial bodies as follows. It has been established that the action of the circumsolar force upon all the Planets (reduced to equal distances) is as the matter in the Planets. The same is similarly established concerning the action of the circumjovial force upon the Satellites of Jupiter; and the same is true of the attraction of all the Planets towards each one. And it follows from this, by Prop. 39, that the attractive force of each individual planet is as its quantity of matter.

From Separate Manuscript Pages, *MS_x* [Add. 3965.3:7-14]

Prop. 39 (neé 38). If several bodies mutually attract [*trahant*] each other ~~and at equal distances~~ and all the attractions [*attractiones*] [^], broken off as one alone [*dirupto uno*] in whatever place, towards each single body at equal distances are always [^] ~~are individually~~ as the heavinesses of the attracted bodies [*pondera attractorum*], and yet the attractive forces [*vires attractivae*], other things being the same [*ceteris paribus*], are as the attractions, I say that the individual attractive forces [*vires attractivae singularum*] are as the heavinesses [*pondera*] of the bodies themselves.

Scholium. We are directed by these propositions to a consideration [*speculationem*] of the ~~proportion~~ [^] agreement [*analogiam*] [^] between centripetal forces and the central bodies to which those forces are usually directed. **For it accords with reason [*rationali enim consentaneum*] that forces which are directed to bodies should depend on the nature and quality of those bodies, as happens in the case of magnetic ones,** and hence ~~should be compounded of innumerable smaller forces tending to the individual attractive particles of bodies.~~ Accordingly, since we have so far considered [*speculati*] forces tending but to a single mathematical point set at the center of each attracting body, it will now be appropriate [*e re erit*] to set out [*exponere*] in brief the circumstances and also the laws of forces composed of lesser forces tending to the individual particles of bodies. [^] And whenever cases of this sort occur the attractions of bodies are to be reckoned [*aestimandae erunt*] by assigning to their individual particles each its own force and gathering [*colligendo*] the sums of these forces. Let us therefore see, where spherical bodies are composed of attractive particles, with what forces these are bound [*debeant*] to act mutually among themselves [*in se mutuo*] and what motions ensue therefrom. [^]

Newton's Third Law of Motion

In the late 1670s

Problem XII: Having given the Magnitudes and Motions of Spherical Bodies perfectly elastic, moving in the same right Line, and striking against one another, to determine their Motions after Reflexion.

The Resolution of this Question depends on these Conditions, **that each Body will suffer as much by Reaction as the Action of each is upon the other, and that they must recede from each other after Reflexion with the same Velocity or Swiftnes as they met before it.** These things being supposed, let the Velocities of the Bodies A and B be *a* and *b* respectively; and their Motions (as being composed of their Bulk [*mole*] and Velocity together) will be *aA* and *bB*....

In early 1685 (...mediis regulariter...)

Law 3. As much as every body acts on another so much does it suffer [*pati*] in reaction. ^ Whatever presses [*premit*] or pulls [*trahit*] another thing is by this pressed or pulled so much. ^ If a bladder full of air presses or carries another entirely similar to it both yield equally inwards. If a body impinging on another by its force changes the motion of the other, then its motion (because of the equal mutual pressure) is so much changed by the force of the other. If a loadstone pulls [*trahit*] iron it is in turn pulled so much, and thus in other [cases]. ^ **In fact this Law is determined [*constat*] by Defs. 12 and 14 in so far as the force exerted by a body to conserve its state would be [*sit*] the same as the force impressed on the other body changing its state, and the former force would be [*sit*] proportional to the change of state of the former and the latter, to the latter.** ^

refertur ad ~~motum~~ ~~non~~ ~~compositum~~. hinc et alia viri in corpore
elasticitate, mobilitate, leuitate et prudentia quae dicuntur considero.

Def. 10. Momenta quantitatum sunt ipsarum principia gene-
-rantia vel alterantia fluxu continuo: ut tempus praesens praes-
-enti et futuri, motus praesens praesenti et futuri, vis centrifuga
aut alia quavis momentanea impetus, punctum linea, linea super-
-ficii, superficies solidi et angulus contactus anguli rectilinei.

Def. 13. Exponentes temporum spatorum motuum claritate
et virum sunt quantitates a vis proportionales exponendis

haec omnia fecit explicare usum est ut Lector, ^{praesens} ~~aliam~~ et distinctis
principiorum ~~mechanicorum~~ ^{mechanicorum} conceptibus imbutus accederet ad sequentia
Quantitati autem absolutae et relativae ab invicem ~~accideret~~ ^{sedulo} distingu-
-ri ~~non potest~~ ^{non potest} eo, quod phenomena omnia pendant ab absolutis,
Vulgus autem qui cogitationes a sensibus abstrahere nesciunt
semper loquuntur de relativis, usque adeo ut absurdum foret vel
sapientibus ~~aut~~ ^{vel etiam} prophetis apud hos aliter loqui. Unde et sacra liti-
-ra et scripta Theologorum de relativis semper intelligenda sunt,
et crasso laboraret praesidio qui inde de rerum naturalium
motibus absolutis ^{philosophis} disputationes moueret. ~~Primum est ac si qui~~
~~duos in magnitudine non apparentes ad absolute ~~accideret~~ inter duo~~
~~maxima lumina ~~quorum~~ ~~motum~~ ~~contradict.~~ ~~quarta~~~~

Leges motus.

Lex 1. Vi impicta corpus ^{omne} ~~statim~~ perseverare in statu suo quies-
-cendi vel movendi uniformiter ^{in linea recta} nisi, quatenus visus
impictis ~~motu~~ ~~impictis~~ cogitur statum illum mutare. Motus autem

Lex 2. Mutationem motus proportionalem esse vi impictae et fieri
secundum lineam rectam quae vis illa imprimitur. Haec duas legibus
jam receptissimis Galilei novit projectilia gravitate uniformiter et secundum lineas parabolae

Lex 3. Corpus omne tantum pati reactione quantum agit in
alterum. ~~Lex 3a. Si repica aere plena promit vel firit alteram~~

sibi consimilem cedet utraq equaliter introsum. ~~Si repica tra-~~
~~hit ferens ipse ~~in ipso~~ ~~totum~~ ~~trahitur~~~~. Si corpus impingens in
alterum vi sua mutat motum alterius et ipsius motus (ob equali-
-tatem pressuris mutuae) vi alterius tantum mutabitur. Si magnae
trahit ferrum icche missim tantum trahitur, et sic in aliis. ^{Consid}
vix haec Lex per Def. 12 et 14 in quantum vi corporis id statu sui conservacionem

Lex 4. Porphyreum dato spatio inclusorum eodem esse motus
inter se sine spatium illud absente quiescat sine moveat id perpetuo

status praesens positionis
velut
petitionis
quam corpus cetero suo agnoscit
solummodo
petitionis
Et subrogatur experientia vis quae
Parabolae
describitur.
Ab
motus projectionis ab resistunt ams
circulans circa axem
parabolae
motus projectionis ab resistunt ams

Verifying the Third Law

Determine two ratios:

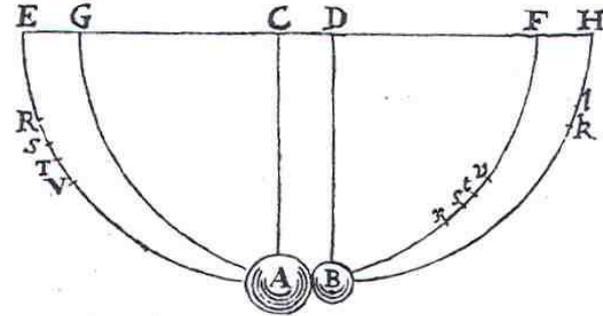
$\Sigma m_i v_i$ after impact

$\Sigma m_i v_i$ before impact

Speed of separation

Speed of approach

Latter now called the
coefficient of restitution



I have tested this as follows with tightly wound balls of wool strongly compressed. First, releasing the pendulums and measuring their reflection, I found the quantity of elastic force; then from this force I determined what the reflections would be in other cases of their collision, and the experiments matched. The balls always rebounded from each other with a relative velocity that was to the relative velocity of their colliding as around 5 to 9. Steel balls rebounded with almost the same velocity and cork balls with a slightly smaller velocity, while with glass balls the proportion was around 15 to 16. **And in this manner, the third law of motion, insofar as it relates to impacts and reflections, is proved by this theory, which plainly agrees with experiment.**

25. *The forces decrease from the surfaces of the planets outwards in the duplicate ratio, and inwards in the ratio of the distances from the centers.*

Therefore, just as the parts of the Earth attract one another, so also do the parts of the planets. If Jupiter and its Satellites were to come together and be formed into a single globe, they would individually doubtless continue to attract [*trahere*] each other as before; and conversely, if the body of Jupiter were resolved into several globes, it must be believed that these would then attract [*traherent*] one another no less than they now attract [*trahunt*] the Satellites. It comes about by these attractions that the bodies of the earth and all the Planets affect a spherical shape and that their parts cohere and are not scattered through the aether. **It has already been established that these forces arise from the universal nature of matter, and therefore that the force of a whole globe is composed of the forces of all the particles of which it is made.** It follows from this (by Prop. 44, Corol. 3) that the force of each particle decreases in the duplicate ratio of the distance from that particle and (by Prop. 43 and 45) that the force of the whole globe decreases outwards from its surface in the duplicate ratio of the distances from the center, and decreases inwards in the simple ratio of the distances from the center, provided that the globe consists of uniform matter. And even if globes are not uniform in proceeding from the center to the circumference, the decrease in the duplicate ratio of the distance outwards will nevertheless hold (by Prop. 46), provided that the non-uniformity is similar everywhere in proceeding around; and two globes of this sort, by the same Proposition, will attract each other by a force decreasing in the duplicate ratio of the distance between their centers.

26. The quantities of the forces and the quantities of the motions arising from them in individual cases.

The absolute force of each globe is therefore as the quantity of matter in it. And the collective-*[collectiva]* motive *[motrix]* force by which one globe is drawn *[trahetur]* toward the other one, and which people commonly designate in terrestrial bodies by the word weight *[ponderis]*, is as the product of the quantities of matter in the two globes divided by the square of the distance between their centers, by Prop. 46, Corol. 4; and the quantity of motion with which each of the two globes will move in a given time toward the other is proportional to this force; and the local *[localis]* accelerative *[acceleratrix]* force by which one globe is attracted *[atrahitur]* toward the other in proportion to its matter is as the quantity of matter in the other globe divided by the square of the distance between their centers (by prop. 46, Corol. 2); and the velocity with which the attracted globe will move in a given time towards the other is proportional to this force. Now that these things have been properly understood, it will be easy to determine the motions of the celestial bodies **among themselves** *[inter se]*.

De Motu Corporum, Definitiones [Add. 3965.5:21]
(a draft in Newton's hand)

1. *Quantity of matter* is that which arises from its density and magnitude [*magnitudine*] jointly. A body of a double density in a double space is quadruple. This quantity I designate under the name of body [*corporis*] or mass [*massae*].

3. *Inherent force of matter* [*Materiae vis insita*] is the [^]~~inertia or~~[^] power of resisting by which a body continues, as much as in it lies [*quantum in se est*], in its state either of rest or moving uniformly straight forward. And it is proportional to its body [^]nor does it differ at all from the inertia of the mass [*inertia massae*] except in our way of conceiving it.[^] In truth a body only exerts this force in the change of its state made by means of another force impressed on it and ~~in its exertion the Resistance consists. This, then, is proportional not indeed to the resisting body but to the impressed force: for the exertion of the innate force to preserve a state is as the force impressed on it in the change.~~ [^]and the Exercise of it is Resistance and Impetus, which are distinct from one another [*ab invicem*] in only one respect: it is Resistance insofar as the body ~~to conserve its own motion~~ struggles [*reluctatur*] against the impressed force, Impetus insofar as the body, not easily yielding, may endeavor to change the state of another body. Common folk, moreover, ~~more usually~~ attribute resistance to bodies at rest, and impetus to those in motion: ~~but rest as it is commonly conceived is not true rest~~ but motion and rest as commonly conceived are distinguished from one another only by point of view [*respectu solo*], nor are [bodies] truly at rest that are commonly considered at rest.[^]

4. *Impressed force* [*Vis impressa*] is ~~what exerts itself~~ [*nititur*] ~~to disturb a body from~~ [^]an action exerted on a body to change[^] its state either of rest or moving uniformly straight forward[^]. This force consists in that action alone, and does not remain in the body after the action. It is moreover of diverse origins[^], as from impetus, from pressure, ~~from resistance~~, and from centripetal force.

5. *Centripetal force* [*Vis centripeta*] is either an action or a power [*potentia*], whichever you please [*quaelibet*], by which a body is drawn [*trahitur*], impelled, or in any way tends toward some point as if to a center. Of this kind is the gravity [*gravitas*] by which a body tends to the center of the earth, the magnetic force by which iron seeks the center of a magnet, and the force, whatever it may be, by which Planets are retained in their orbits and perpetually constrained lest they go off in their tangents. Centripetal force moreover is three-fold in its quantity: ***absolute, accelerative, and motive***. The *absolute quantity* (which can also be called *absolute force*) is greater to one center and less to another, without respect to the distances and magnitudes of the attracted [*attractorum*] bodies; as the magnetic strength is greater in one magnet, less in another. The *accelerative quantity* or *force* is as proportional to the velocity which it generates in a given time; as the strength of the same magnet is greater at a lesser distance and less at a greater, or the gravitational force [*vis gravitans*] is greater close to the earth, less in higher regions. The *motive quantity* or *force* is as proportional to the motion which it produces in a given time; as weight [***pondus***] is greater in a larger body and less in a smaller one. So therefore *motive force* is to *accelerative force* as *motion* is to *speed* [*celeritatem*]. For the *quantity of motion* arises from the *speed* multiplied into the moving body, and the *quantity of motive force*, from the *accelerative force* multiplied into the same body. **Whence, near the surface of the earth where accelerative gravity is the same in bodies universally [*universalis*], motive gravity or weight [*pondus*] is as the body; but, in receding further from the earth and ascending to regions where accelerative gravity comes to be less, weight will be correspondingly diminished and will be always as the body multiplied into the accelerative gravity.** *Attractions* and *impulses* I name *accelerative* and *motive* in the same sense. Moreover, the words *attraction*, *impulse*, or any *propensity* to a center I employ indifferently and interchangeably [*pro se mutuo*], considering these forces not *physically* but merely *mathematically*. Whence the Reader should beware lest he think I anywhere define the kind or mode of an action, or its physical cause or reason.

From *De Motu Corporum, Liber Primus* [Dd. 9.46]
Definitiones

1. [^]Quantity of matter is a measure of the same arising from its density and magnitude jointly. Air, if the density is doubled and the space is doubled, is quadrupled. Understand the same for snow and powder condensed by compression or liquefaction. And it is the same for all bodies that are diversely condensed by operations of nature. I here legitimately do not take into account any medium, if there should be any, freely pervading the interstices between the parts. The quantity of matter becomes known always from the weight [*pondus*] of a body. For, by experiments with pendulums carried out most accurately, I have found it to be proportional to the weight as will be set out below. The same quantity in fact I understand under the name body or mass in the following pages.[^]
[N.B. drafted by Newton on the verso side]

{**Canceled:** 2. The axis of matter is an ever so straight line around which the matter, kept in the parts located among themselves, is able to revolve [^]uniformly[^] in a space free of impediment and incitement [*incitamentis*].

3. The center of matter is the ~~eoneurrence~~[^]intersection[^] of two axes and in a uniform body is what is commonly called the center of gravity. Yet even in non-uniform matter it is the same as the center of gravity if the manner in which that center is determined is not from the size [*magnitudine*] but from the quantity of matter. For example, the center of gravity by size of a gold globe and a wooden globe is that point which divides the distance between the centers of the globes in the ratio of their sizes, but the true center of gravity, and the center of matter, is that point which divides this distance in the reciprocal ratio of either the weights [*ponderum*] or the quantities of matter of the globes. For the matter in the bodies individually I have found from experiments with pendulums to be proportional to the weights [*ponderi*], as will be set out below.
canceled}

5. *Centripetal force* is that by which bodies are drawn [*trahitur*], impelled, or otherwise tend toward some or other point as if to a center....
6. The *absolute quantity of a centripetal force* is the measure of the same that is greater or less in proportion to [*pro*] [^][*ratione causa fortius a*] [^] the efficacy of the cause [*propagantidi virae*] [^] propagating it [*eam propagantis*] [^] from the center through surrounding regions [*regiones in circuitu*]: as the magnetic strength is greater in one magnet than in another.
7. The *accelerative quantity of a centripetal force* is the measure of it proportional to the velocity it generates in a given time. As The strength of the same magnet is greater at a lesser distance, less at a greater; the gravitational force is greater in valleys, less at the peaks of high mountains (as established by pendulum experiments), and still less (as will be made clear below) at great distances from the Earth, yet is everywhere the same at equal distances, because it equally accelerates all falling [*cadentia*] bodies (heavy [*gravia*] or light, large or small), absent the resistance of air [*sublata aeris resistantia*].
8. The *motive quantity of a centripetal force* is the measure of it proportional to the motion it generates in a given time. As for instance weight [*pondus*] is greater in a larger body and less in a smaller body, and in one and the same body is greater near the earth and less out in the heavens. This force is the total centripetency or propensity [*centripetentia seu propensio*] of a body [^]toward a center [^] and (so to speak) its weight [*pondus*] ~~toward a center~~, **and may always be known from the force opposite and equal to it that can prevent a body from falling.**

These quantities of forces, for the sake of brevity, may be called **absolute, accelerative, and motive forces** and for the sake of differentiation be referred to a body, a body in a place, and a center of force; that is, motive force to a body as an endeavor and propensity of the whole toward a center compounded from the propensities of all the parts, and accelerative force to a place of a body as a certain efficacy diffused from the center through each of the surrounding places in order to move bodies that are in them; absolute force, moreover, to a center ~~or a kind of force toward some body existing in a center~~, ~~as an efficacy to propagate accelerative forces from it throughout all the surrounding regions.~~ This concept is mathematical. For I now ignore the physical causes of forces.

The Genesis of Newton's Numerator, The Evidence from the Manuscripts

- **Supports Stein's conjecture that "this whole conception of the constitutional frame of nature" was developed "at the same time that he was discovering the law of gravity"**
- **The key factor driving the two: the half of the numerator that was indeed derived from phenomena, *viz.* the distinctive feature of gravity, its "proportioning itself" to each body on which it acts**
- **1685, "a year of gestation": development of a vocabulary and a way of framing this "conception of the frame of nature" that would enable him to present it and the law compellingly**
- **Throughout 1685 Newton was fully aware that "it is still not decided by any single phenomenon that the attractive forces of heavenly bodies are proportional to their masses"**
- **In writing Book 3 during the latter half of 1686, he chose to suppress this lacuna by noting merely parenthetically in a corollary preceding the question of the numerator that Law 3 holds for celestial gravity**

Late 1685: A Proposed Experimental Approach to Verifying that Gravity Varies as $M_{\text{ATTRACTING}}$

Proposition 92: Given an attracting body, it is required to find the ratio by which the centripetal forces tending toward each of its individual points decrease:

From the given body a sphere or cylinder or other regular figure is to be formed, whose law of attraction corresponding to any ratio of decrease is to be found by Props. 80, 81, and 91. **Then, by making experiments, the force of attraction at different distances is to be found; and the law of attraction toward the whole that is thus revealed will give the ratio of the decrease of the forces of the individual parts, which was required to be found.**

From Results in Section 12:

Stationary large spheres attracting small spheres, to verify

- $1/r^2$ vs. e.g. $1/r$, $1/r^3$, and $1/r^4$ etc. sphere-to-sphere attraction
- Proportionality to mass of large sphere, by varying this mass

From Results in Section 13:

Stationary large cylinders attracting small spheres, to verify

- Dependence on distribution of matter in the attracting body
- And, thereby, $1/r^2$ particle-to-particle attraction
- (Spheroids added in summer of 1686)

PROPORTIONALITY TO $M_{\text{attracting}}$

“The G-determinations of Poynting and of Richarz and Krigar-Menzel are of special value in relation to the question of how far the proportionality of the attractive force to the mass is guaranteed for masses of the same material. Both experimenters used unobjectionable laboratory methods carried out with the greatest care. Both determinations employed the same material (lead) and the same method of measurement, but masses of very different magnitudes (154 or 100,000 kg). Even though in one case the mass was 650 times greater than the other, the results agree to approximately 0.2%.”

Zenneck, “Gravitation,” 1901, *Encyklopädie der mathematischen Wissenschaften*

	Observér	attracting mass	Δ
Laboratory methods	<i>Boys</i>	7.4 kg	5.527
	<i>Braun</i>	9.1 ”	5.5270
	<i>Poynting</i>	154 ”	5.4934
	<i>Wilsing</i>	325 ”	5.577
	<i>Richarz and Krigar-Menzel</i>	100.000 ”	5.5050

		Δ	G
Torsion balance	<i>Boys</i>	5.527	$6.658 \cdot 10^{-8} \text{ cm}^3 \text{ s}^{-2} \text{ g}^{-1}$
	<i>Braun</i>	5.5270 ^a	
(Double pendulum	<i>Wilsing</i>	5.577	$6.596 \cdot 10^{-8} \text{ cm}^3 \text{ s}^{-2} \text{ g}^{-1}$)
Balance	<i>Poynting</i>	5.4934	$6.698 \cdot 10^{-8} \text{ cm}^3 \text{ s}^{-2} \text{ g}^{-1}$
	<i>Richarz and Krigar-Menzel</i>	5.5050	$6.685 \cdot 10^{-8} \text{ cm}^3 \text{ s}^{-2} \text{ g}^{-1}$
Mean value of these determinations		5.513 ^b	$6.675 \cdot 10^{-8} \text{ cm}^3 \text{ s}^{-2} \text{ g}^{-1}$